Modelling of the Road Transport Networks by Mathematical Method Using Lyapunov Function

Asralt BUYANTSOGT[a], Unursaikhan DAMBADORJ[b]

[a],[b] Department of Logistics and Transport Management, School of Mechanical Engineering, Mongolian University of Science and Technology, Ulaanbaatar, Mongolia
[a]*E-mail:* aska_b@yahoo.com
[b]*E-mail:* unur05@yahoo.com

Abstract: The paper introduces a method of mathematical modeling of high scale road traffic networks, where a new special hyper-matrix structure is intended to be used. The structure describes the inner-inner, inner-outer, outer-outer relations, laws of a network area. The research examined the nonlinear equation system of the nonlinear positive system. The asymptotic stability of the system is examined with the method of Lyapunov functions. It gives a solution for the control of this non-autonomic system through Lyapunov functions, which give an eligible solution for the asymptotic stability of the whole region and some regions which need the movement control.

Key words: Lyapunov functions, network, nonlinear equation system

1. INTRODUCTION

To model the road transport network we have to assume the following emphasely:

- Geometrical location, geometrical quantity
- The type of the transprotation and the movement control(traffic light, traffic signsetc)
- The road traffic rules and laws.
- Individual participants and his/her right or responsibility.
- several kinds of outer factors (such as seasonal and weather effects)

The main standart of road transport modelling is the issue of how does the model suit for real life and what the resultis.

- the model must take into consideration all elements, which have a genuine effect throughout the operation of the system and the negligence of which would distort the results
- it must be correctly supported mathematically
- In case of simulation it must be numerically fast. In case of regulation, at least a real-term regulation must be achieved

2. THE RELATION BETWEEN THE CONVENTIONAL, JUNCTION-CENTRIC MAP MODEL AND THE NETWORK-CENTRIC MODEL

The road traffic network models, well-known from specialized literature, treat junctions or intersections as essential elements in models. This results such a graph, which loyally copies maps, the top of mountains are the junctions or intersections, and the springs and rivers which run down the mountainside are the road sections connecting them.

If we take a look at an urban or route map, the map is such a graph, the apices of which are constituted from traffic junctions, and its edges are constituted from the roads connecting the junctions.

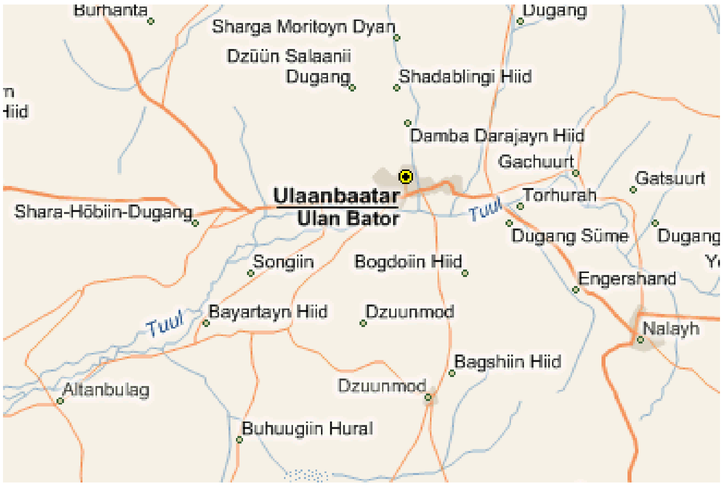


Figure 1 The apieces of the graph are the traffic junctions

Refining the map (*Figure 1.2, Figure 1.3*):

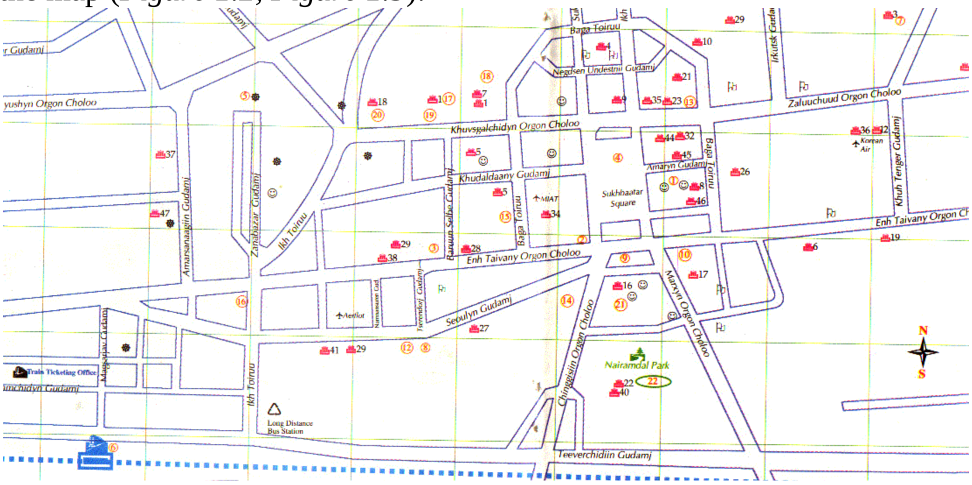


Figure 2. Refined map-graph

To the detailed network, all intersections appear in the set of apices, and the edges are also extended with all the road sections.

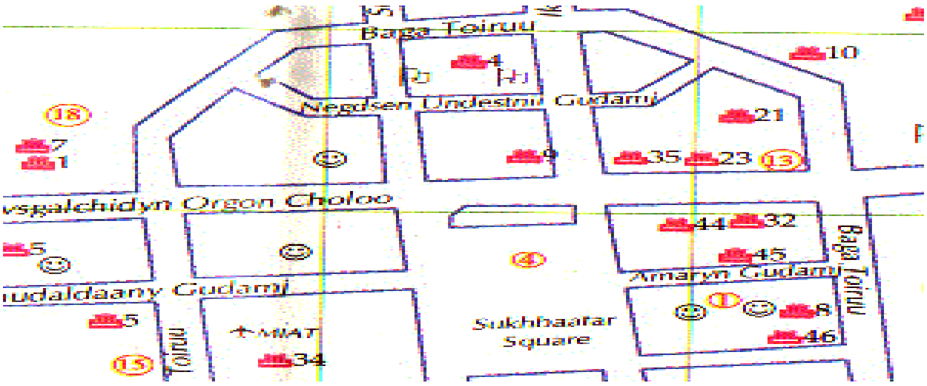


Figure 3. Detailed map-graph

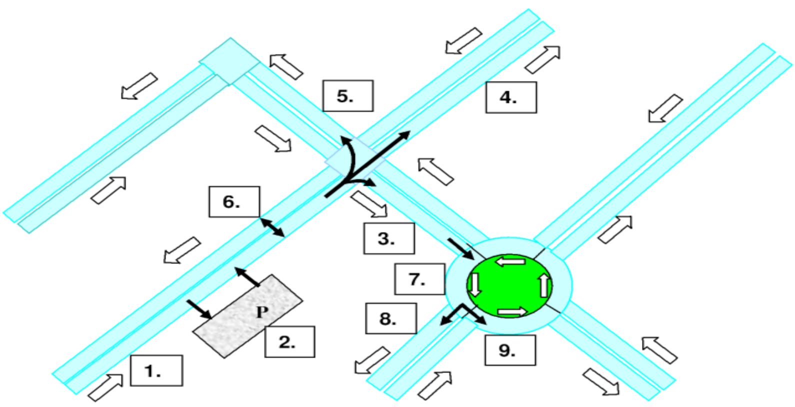
This description, therefore, naturally generated the approach that the central place is taken by the junctions (intersections), as they are the apices of the graph and during the transaction of the traffic the apices co-operate with each other through the road sections connecting them.
It’s fairly extensive and modern research trends appear in the fields of intelligent junctions [5], co-operative junctions – agents, game-theory methods, etc. An especially important field of research is the examination of roundabout junctions [6].
The optimal function of junctions is, therefore, very important in the system. However, if we consider their role more thoroughly, they are the “necessary evil” in the network. It would be ideal from transport’s point of view if less and less intersection traffic was there! What’s more, if they didn’t exist, and one could get from any point to another without intersection traffic!
This obviously sounds absurd, but this thought leads us to another approach. It raises the question whether it is indeed necessary to have junctions as the focus of examinations. The correct answer to this is that as far as transport is concerned, the focus of examination has to be placed on the whole of the network.

3. THE COMPONENTS OF THE NETWORK-MODEL, FEATURES OF CONDITION, REGULATED CO-OPERATIONS DEPENDING ON THE FEATURES OF CONDITION, NETWORK RELATIONS AND THE MATHEMATICAL MODEL

3.1 The Components of the Model.

The components of the network are, as an initial approach, the lanes and the defined parking spaces as well as the parking lanes by the roads. It’s easy to admit that the defined parking spaces and the parking lanes by the roads take part as generalized sections in the function of the network. Therefore, it is the sections that co-operate in the whole network, and these components form the apices of the network graph.

As a simple example, let’s take a look at the functioning and disturbing co-operation of some numbered components as seen on the 4^{th} diagram: The 1^{st} component co-operates with the 2^{nd}, 3^{rd}, 4^{th}, 5^{th} and 6^{th}. The 3^{rd} component co-operates with the 7^{th}. The 7^{th} component co-operates with the 8^{th} and 9^{th}.



1- road for motion transmission
2- parking
3- to turn to the right motion
4- directly motion
5- to turn to the left motion
6- width of road
7- to move in circle motion
8- go out from circle motion
9- allround motion

Figure 4.The components of the network

The edges of the controlled graph are dynamic relations, as the relations between the co-operating apices are dynamic. This relation is described by the relational matrix. It takes into consideration all elements included in the map as well as all rules (including lights as well), which determine how the traffic runs. (The rules specify the traffic on the elements, as well as set conditions on stepping from one element over to another. Our map also contains important parameters, such as the length, width, number of lanes, the number of vehicles that can be placed in parking spaces, numeric figures of the allowed speed, which are considered at the parameters of the dynamical model.)

This model, therefore, analyses the whole network in addition to the complete relation-system. The "junction" does not appear as an individual component here, as the function of each junction is a part of the relation-system.

3.2 Some Features of the Transport Network

- The parallel lanes have an effect on each other. This interaction, which refers to functioning on each other or disturbing each other, influences the density and speed of vehicles occurring on parallel lanes.
- The oncoming traffic also has an effect on the "right" and the "left" lanes. This interaction manifests itself in the disturbance caused by passing each other.

The defined parking spaces as well as the parking lanes by the roads take part as generalized sections in the function of the network, and the vehicles parking there also interact with those network sections and arcs, with which they have a direct traffic relation. This relation of timely altering intensity is capable of creating peak-load in the examined network by itself, for example, without having

- Any traffic arriving from a defined outer network onto this one.
- Inner autonomists of vehicle transfer also function among the relating components of the network. For example, having a green light doesn't result in transfer if the density of traffic is too high on the recipient section or nil on the transmitting section.

3.3 Feature of Condition and its Relation to "Conventional Density"

In our model, the (geometrical) ***vehicle density*** refers to that ***s*** dimension-free ($0 \leq s \leq 1$) figure, which measures the proportion of the total length of vehicles on a certain section and the length of the section. The densities occurring on the road sections of the inner network are the system's features of condition.

Our transport network model, consisting of n pieces of inner sections, describes that route/urban transport system, which is located in a region bordered by a closed curve.
In this case, the (H_i) vehicle densities evolving on the inner network are the **system's features of condition**, in order as $x_1(t), x_2(t), x_3(t), ...x_n(t)$. The model also uses the sub-network of the outer network (H_k), which consists of such m pieces of sections that have a direct relation with some inner section. Vehicle densities evolving on them are indicated by $s_1(t), s_2(t), ...s_m(t)$, which are **known based on measurements**. Our mathematical model depicting the network takes into consideration the inner relations of the network within the region and the outer relations of it outside the region.

Conventional vehicle density or traffic density is interpreted in specialized literature as the number of vehicles being on a given road section, in ***t*** amount of time (NB: this amount, if needed for the study of traffic flow modeling, can be used in a differentiated form as well). Its sign: e.g. ***S***, its measuring unit can be: (number of vehicles/km)

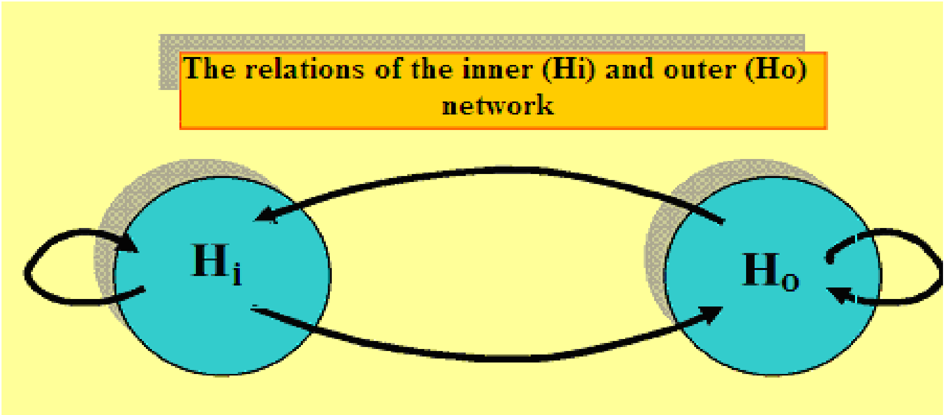


Figure 5. The relations of the inner and outer network

The hereby initiated phenomenon of an ***s*** dimension-free (geometrical) density can indirectly converted into the conventional ***S*** density with the usage of the statistical phenomenon of the h length of the unit-vehicle, measured in meter:

$$s = \frac{S \cdot h}{1000} \quad or \quad S = 1000 \times \frac{s}{h} \tag{1}$$

Therefore, the results of the specialized literature [12] revealing the relation between speed and density can be applied to this s (geometrical) density.

The parking spaces participate as generalized sections in the function of the network. Let's take P_i as a parking space where the space itself is N_i, and there is $n_i(t)$ pieces of vehicles in the parking space at t time. In such situation find the feature of condition in time by the formula:

$$x_i(t) = \frac{n_i(t)}{N_i} \tag{2}$$

to which firstly the following condition applies: $0 \le x_i(t) \le 1$,

Secondly, presuming h length of unit-vehicle: $x_i(t) = \frac{n_i(t) \times h}{N_i \times h}$, and therefore the fictive section $l_i = N_i \times h$ can be defined, on which the vehicles take up $n_i(t) \times h$ length of place.

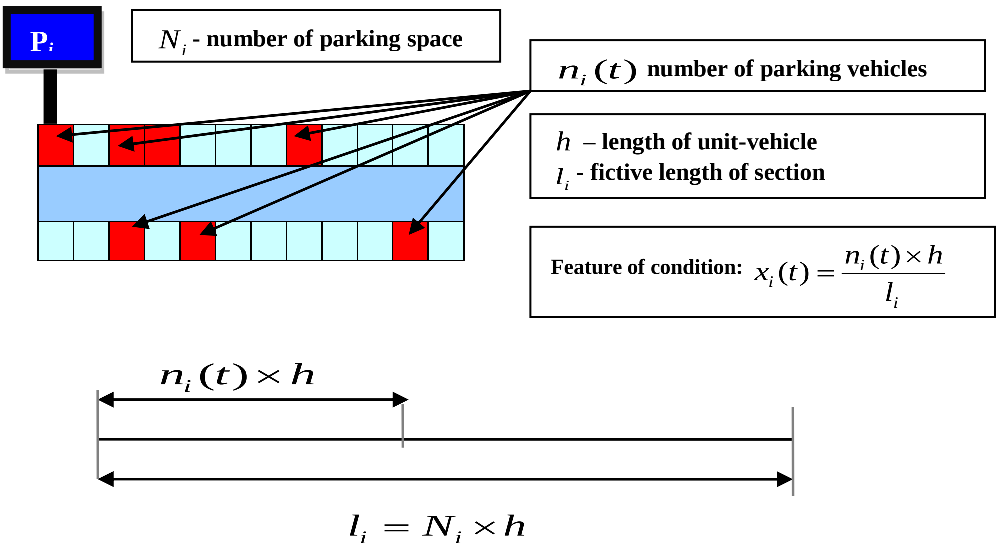
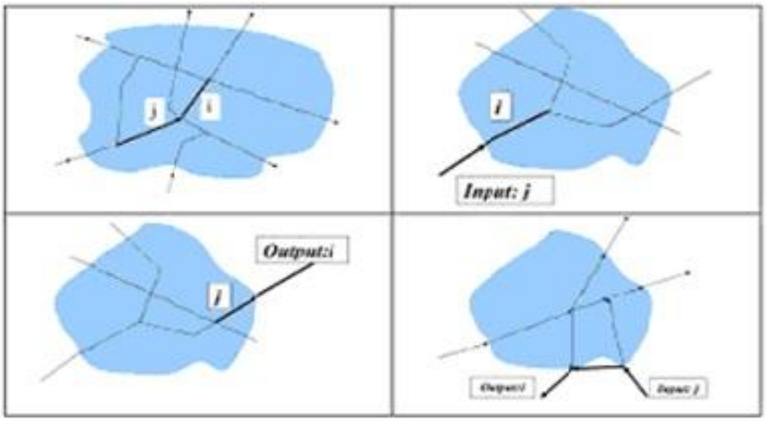


Figure 6. The parking spaces are generalized sections

3.4 Relation of outer and inner traffic networks and the relational matrix

In order to create the mathematical network model, the network-defining relational matrix, which is a hyper-matrix, was of capital importance [1,2,3,4].



$$\begin{matrix} i-\textit{for the inner section} \\ j-\textit{for the outer section} \end{matrix} \begin{bmatrix} \underline{K}_i & \underline{K}_{inp} \\ \underline{K}_{outp} & \underline{K}_o \end{bmatrix}$$

Figure 7. The relational hyper-matrix of the inner and outer system

The inscription of the relational matrices generates a relation where a j. section co-operates with the i. section:

Relation of section number j

(j)

$$\underset{(nxn)}{\underline{K}} = \begin{bmatrix} & \vdots & \\ \ldots\ldots K_{ij} = k_{ij}(t)S_j(t)E_j(t)v_{ij}(t)\ldots \\ & \ldots & \\ & \vdots & \end{bmatrix} \ldots\ldots\ldots\ldots\ldots \quad \textit{with section } i\ldots(i) \tag{3}$$

The analyzed model can be applicable to the simulation test and planning of large-scale road traffic networks, and to the regulation of traffic systems. The hyper-matrix of the 7th diagram means the following in details:

$\begin{bmatrix} 0 & k_{1,2}(x_1,x_2) & \ldots & k_{1,j}(x_1,x_j) & \ldots & k_{1,n}(x_1,x_n) \\ k_{2,1}(x_2,x_1) & 0 & \ldots & k_{2,j}(x_2,x_j) & \ldots & k_{2,n}(x_2,x_n) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ k_{j,1}(x_j,x_1) & k_{j,2}(x_j,x_2) & \ldots & 0 & \ldots & k_{j,n}(x_j,x_n) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ k_{n,1}(x_n,x_1) & k_{n,2}(x_n,x_2) & \ldots & k_{n,j}(x_n,x_j) & \ldots & 0 \end{bmatrix}$	$\begin{bmatrix} Ki_{1,1}(x_1,s_1) & Ki_{1,2}(x_1,s_2) & \ldots & Ki_{1,j}(x_1,s_j) & \ldots & Ki_{1,m}(x_1,s_m) \\ Ki_{2,1}(x_2,s_1) & Ki_{2,2}(x_2,s_2) & \ldots & Ki_{2,j}(x_2,s_j) & \ldots & Ki_{2,m}(x_2,s_m) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ Ki_{j,1}(x_j,s_1) & Ki_{j,2}(x_j,s_2) & \ldots & Ki_{j,j}(x_j,s_j) & \ldots & Ki_{j,m}(x_j,s_m) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ Ki_{n,1}(x_n,s_1) & Ki_{n,2}(x_n,s_2) & \ldots & Ki_{n,j}(x_n,s_j) & \ldots & Ki_{n,m}(x_n,s_m) \end{bmatrix}$
$\begin{bmatrix} K_{01,1}(s_1,x_1) & K_{01,2}(s_1,x_2) & \ldots & K_{01,j}(s_1,x_j) & \ldots & K_{01,n}(s_1,x_m) \\ K_{02,1}(s_2,x_1) & K_{02,2}(s_2,x_2) & \ldots & K_{02,j}(s_2,x_j) & \ldots & K_{02,n}(s_2,x_n) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ K_{0j,1}(s_j,x_1) & K_{0j,2}(s_j,x_2) & \ldots & K_{0j,j}(s_j,x_j) & \ldots & K_{0j,n}(s_j,x_n) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ K_{0m,1}(s_m,x_1) & K_{0m,2}(s_m,x_2) & \ldots & K_{0m,j}(s_m,x_j) & \ldots & K_{0m,n}(s_m,s_n) \end{bmatrix}$	$\underline{\underline{K}}_o$

Figure 8. The relational hyper-matrix in details

The $\underline{\mathbf{K}}$ constructed matrix made up of the $\underline{\mathbf{K}}_i$ and the $\underline{\mathbf{K}}_o$ matrices appears in the differential-equation system acting as the mathematical model:

$$\begin{bmatrix} -\left(\sum_{i=1}^{n} k_{i,1}(x_i,x_1)\right)-\left(\sum_{i=1}^{m} K_{0i,1}(s_i,x_1)\right) & k_{1,2}(x_1,x_2) & \cdots & k_{1,j}(x_1,x_j) & \cdots & k_{1,n}(x_1,x_m) \\ k_{2,1}(x_2,x_1) & -\left(\sum_{i=1}^{n} k_{i,2}(x_i,x_2)\right)-\left(\sum_{i=1}^{m} K_{0i,2}(s_i,x_2)\right) & \cdots & k_{2,j}(x_2,x_j) & \cdots & k_{2,n}(x_2,x_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{j,1}(x_j,x_1) & k_{j,2}(x_j,x_2) & \cdots & -\left(\sum_{i=1}^{n} k_{i,j}(x_i,x_j)\right)-\left(\sum_{i=1}^{m} K_{0i,j}(s_i,x_j)\right) & \cdots & k_{j,n}(x_j,x_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{n,1}(x_n,x_1) & k_{n,2}(x_n,x_2) & \cdots & k_{n,j}(x_n,x_j) & \cdots & -\left(\sum_{i=1}^{n} k_{i,n}(x_i,x_n)\right)-\left(\sum_{i=1}^{m} K_{0i,n} \right. \end{bmatrix}$$

Figure 9. The $\underline{\mathbf{K}}$ constructed matrix

3.5 The Mathematical Model.

When inscribing the mathematical model, the below first-rate nonlinear differential equation system was generated for the feature of condition-vector of the inner sections $\underline{x}(t)$ [1,2,3].

$$\underline{\mathbf{x}}(\mathbf{t})'_{(\mathbf{n}\times\mathbf{l})} = <\underline{1/l_i}>_{(\mathbf{n}\times\mathbf{l})} \left[\underline{\mathbf{K}}_{(\mathbf{n}\times\mathbf{l})}\underline{\mathbf{x}}(\mathbf{t})_{(\mathbf{n}\times\mathbf{l})} + \underline{\mathbf{K}}_{\mathbf{inp}(\mathbf{n}\times\mathbf{l})}\underline{\mathbf{s}}(\mathbf{t})_{(\mathbf{m}\times\mathbf{l})}\right] \quad (4)$$

Where $<\underline{1/l_i}>$ is a diagonal matrix containing the reciprocal figures of the inner sections' length, the components of the $\underline{K}(x(t),s(t))$ and $\underline{K}_{inp}(x(t),s(t))$ relational matrices contain the relational functions and functions dependant on density conditions, the physical meaning of the components is speed. The system is a positive system, the model is essentially a macroscopic model.

4. STABILITY AND CONTROL ACHIEVED THROUGH THE LYAPUNOV-STABILITY

Let's consider the (4) differential equation system in a slightly more concise form:

$$\underline{\mathbf{x}}' = <\underline{\underline{L}}>^{-1}\left[\underline{\mathbf{K}}(\underline{\mathbf{x}},\underline{\mathbf{s}})\underline{\mathbf{x}} + \underline{\mathbf{K}}_{\mathbf{inp}}(\underline{x},\underline{\mathbf{s}})\underline{\mathbf{s}}\right] \quad (5)$$

It is true for the feature of condition that $0 \le x_i \le 1\ (i = 1,2,...n)$

Let's introduce the $V(x_1,x_2,...x_n) = l_1 \times x_1 + l_2 \times x_2 + ... + l_n \times x_n$ function, in which $0 < l_i$ 0<l_i, az x_i refers to the length of section belonging to the feature of condition.

Briefly, the scalaric product of $\underline{L} = [l_1, l_2, ... l_n]$ $\underline{\mathbf{L}}$ =[l_1, l_2, … l_n] and $\underline{x}$:

$$V(x_1,x_2,...x_n) = \underline{L} \times \underline{x} \quad (6)$$

The $V(x)$ scalaric vector function is positive definite, because:

$V(x) = 0$, only if $x = 0$

$V(x) > 0$, for all non-nil **x** feature of condition in the domain region.

Constituting the W function:

$$W = \frac{\partial V(x)}{\partial t} = (\partial V / \partial x_1)(\partial x_1 / \partial t) + ... + (\partial V / \partial x_n)(\partial x_n / \partial t) = l_1 \times \partial x_1 / \partial t + l_2 \times \partial x_2 / \partial t + ... + l_n \times \partial x_n / \partial t = \underline{L} \times \underline{X'}$$
(7)
Substituting equation (5):
$W = [\underline{\underline{K}}(\underline{x}, \underline{s})\underline{x} + \underline{\underline{K}}_{inp}(\underline{x}, \underline{s})\underline{s}]$
Therefore W is the total of the units appearing on the right hand side of the equation system's equations, at which the Ii section-length parametres drop out due to the multiplication with the diagonal matrix $< \underline{L} >^{-1}$ (The $< \underline{L} >^{-1}$ diagonal matrix contains the reciprocal figures of the section-lengths in its main diagonal.)
Due to the construction of the $\underline{\underline{K}}(\underline{x}, \underline{s})$ relational matrix, after the aggregation all components of $\underline{\underline{K}}_i$ drop outof the functions constituting the W function (as for the x_i coefficients, it is only the components in the $\underline{\underline{K}}_{outp}$ matrix, whereas at the s_i coefficients it is only the components in the $\underline{\underline{K}}_{inp}$ matrix that occur).

$$W = -\left(\sum_{i=1}^{m} ao_{i,1}(s_i, x_1)\right)x_1 - \left(\sum_{i=1}^{m} ao_{i,2}(s_i, x_2)\right)x_2 - \left(\sum_{i=1}^{m} ao_{i,n}(s_i, x_n)\right)x_n + \left(\sum_{i=1}^{n} ao_{i,1}(s_1, x_i)\right)s_1 + \left(\sum_{i=1}^{n} ao_{i,2}(s_2, x_i)\right)s_2 + \left(\sum_{i=1}^{n} ao_{i,n}(s_m, x_i)\right)s_m$$
(8)

The system, therefore, is stabel if the inward transportation is larger on the margins than the outward transportation.
$$\left(\sum_{i=1}^{n} ai_{i,1}(s_1, x_i)\right)s_1 + \left(\sum_{i=1}^{n} ai_{i,2}(s_2, x_i)\right)s_2 + \left(\sum_{i=1}^{n} ai_{i,n}(s_m, x_i)\right)s_m < \left(\sum_{i=1}^{m} ai_{i,1}(s_i, x_1)\right)x_1 + \left(\sum_{i=1}^{m} ai_{i,2}(s_i, x_2)\right)x_2 + \left(\sum_{i=1}^{m} ai_{i,n}(s_i, x_n)\right)x_n$$
(9)
Briefly:
$$\sum F_{input} < \sum F_{output}$$
(10)
The autonom system, however, is always stabil, as it is:
$s_1 := 0 \;\; s_2 := 0 \;\; s_m := 0$
$$W = -\left(\sum_{i=1}^{m} ao_{i,1}(s_i, x_1)\right)x_1 - \left(\sum_{i=1}^{m} ao_{i,2}(s_i, x_2)\right)x_2 - \left(\sum_{i=1}^{m} ao_{i,n}(s_i, x_n)\right)x_n \quad (11)$$

as the velocities appearing in the summaries are not negative.

4.1 For the Physical Meaning of the LyapunovFunction:

Let's examine the physical meaning of the $V(x_1, x_2, ... x_n) = l_1 \times x_1 + l_2 \times x_2 + ... + l_n \times x_n$ function.
According to our definition:
$$x_i = \frac{n_i \times h}{l_i}$$
where: n_i is the number of vehicles on section no. i
h unit-vehicle length

$\frac{n_i \times h}{l_i}$ after replacement:

$$V = (n_1 + n_2 + ... + n_n) \times h$$

is proportionate with the number of all vehicles in the region. More precisely: V generates the total length of road occupied by the vehicles in the inner road network at a given t time.
Therefore, the negative value of the derivate of V(t), according to t, means the decrease of the total number of vehicles as well as the decrease of the total amount of occupied road length in the inner road network.
This (10) examination generates a solution for the control law through the Lyapunov function, which gives an eligible solution for the asymptotic stability of the system and can dynamically applied to the whole region and to its critical sub-domains.

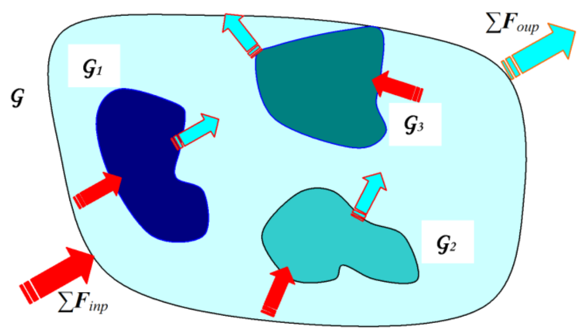


Figure 10. The control law practicing the Lyapunov function in the region and sub-region

5. SUMMARY

The analyzed model can be applicable to the simulation test and planning of large-scale road traffic networks, and to the regulation of traffic systems.
A special macroscopic model has been used, due to which the mathematical model leading to partial differential equation systems can be avoided.

- Junction does not play a quintessential role in our special model! There are sections that either co-operate or not. (For example, a parking space is a special section too, and two parallel lanes can also co-operate.)
- In our model vehicle density refers to the proportion of the total length of vehicles on one certain section and the length of the section.
- Our road traffic model examines the vehicle density, occurring due to the flow, on the sections of the road network, which is located in a region bordered by a closed curve.
- The vehicle processes flowing in and out of the region are considered known. At first glance, these traffic processes are the “inputs” and “outputs” of the traffic system.
- In fact, these (the vehicle densities measured on the inward road sections outside the region, as excitation, and the vehicle densities measured on the outward road sections outside the region, as repression) together form the real input processes of the mathematical model.
- The $x_i(t)$ densities appearing on the road sections of the region are the system’s features of condition.
- A traffic system model consisting of an n inner and m outer road section is used.
- All road sections and parking spaces, which must be considered, are numbered in this region in accordance with the map.

- In order to create the mathematical model, the system-defining relational matrices are of capital importance (*7^{th} diagram*). Our model applies four relational matrices.
- Finally, a nonlinear network model is examined. (*1*)

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