Estimation of Time-Dependent Origin-Destination Matrices Using the Path Flow Proportion Method

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Abstract: This research formulates the time-dependent origin-destination (O-D) matrix estimation problem as a system of linear equations and solves this problem by using the conditional inverse matrix theory. One of the unique aspects of the adopted matrix inverse method is that it provides a generalized matrix inverse procedure even if the target matrix is either singular or non-squared. Due to the multiple solutions problem when solving the O-D matrix estimation problem, the path flow proportion method is developed in the present study and the unique solution of the O-D matrix estimation problem can be obtained. In the numerical analysis, the developed model framework and solution algorithm are evaluated based on a simplified network. The numerical analysis result reveals that the time-dependent O-D demand estimates given by the proposed models and adopted solution algorithm can be estimated exactly.

Keywords: time-dependent origin-destination matrix, path flow proportion method, conditional inverse matrix

1. INTRODUCTION

A trip origin-destination (O-D) matrix is one of the crucial components in network modeling because it describes trip distributions and travel patterns among a set of traffic zones in a vehicular network. From the perspective of transportation planning, an O-D demand matrix contains trip makers' travel directions, route selections, and trip lengths. On the other hand, for the purpose of traffic engineering and/or operation, a time-dependent O-D matrix is one of the key factors to determine an optimal traffic control scheme that achieves some system-wide objectives. Therefore, a trip O-D matrix in a given network is one of the essential inputs to transportation planning procedure and/or traffic control or management.

Traditionally, trip O-D tables in a given vehicular network are obtained via users' surveys such as household survey, roadside interview, or license plate recording, which is very costly and might confront with problems of sampling bias or data recording errors. As the rapid development of intelligent transportation systems (ITS), such a trip O-D matrix can be directly obtained or indirectly estimated in light of the link and/or path flow information provided by some advanced sensor technologies; avoiding the problems associated with traditional O-D data collection methods. In general, the link traffic flows of a given network are the spatial trip distributions among a set of O-D or path flows. These link flows and traffic related parameters could be collected via some advanced sensor technologies in a cost-effective manner, such as pass-type Vehicle Detector (VD) and active-type Automatic Vehicle Identification (AVI) sensors. Thereby, network O-D matrices may be inferred from traffic flow information contained in a set of collected link traffic counts and other parameters (e.g., travel speed, occupancy, turning proportion, etc.) using some suitable methodological approaches.

This research formulates the time-dependent O-D matrix estimation problem as a system of linear equations and solves this problem by using the path flow proportion method and it is solved by the conditional inverse matrix (CIM) theory. The adopted matrix inverse method provides a generalized matrix inverse procedure even if the target matrix is either singular or non-squared. In addition, the issue of multiple solutions generally found in the network O-D demand estimation problem is discussed and solved by the path flow proportion method where time-dependent path flow proportions between each O-D pair are assumed to be known. The present study provides a more general model framework than most of the traditional methods for the network O-D demand estimation problem. The developed models and adopted solution algorithms can be applied to a time-dependent traffic management scheme by providing highway users with desirable routing suggestions in light of the potential trip distributions over a period of time.

The purposes of this research are twofold. First, this research formulates the spatio-temporal relationship between a set of observed link flows and unknown O-D flows via a time-space network structure. Second, the problem is solved using the CIM and path flow proportion methods under various link flow distributions. The remainder of this paper is organized as follows. In section two, a comprehensive investigation of the related literature is conducted and commented on. Sections three and four respectively describe the models for the network O-D demand estimation problem and solution algorithms. Numerical analysis in terms of the experimental setup and model evaluation results are presented in section five. Finally, in section six, findings and conclusions of this research are summarized, and future research directions are also suggested.

2. LITERATURE REVIEW

Inferring network O-D demand matrices by link flow measurements is essential and straightforward in view of the close relationships between observed link/path flows and a set of unknown O-D flows. Robillard (1975) proposed the pioneer idea of inferring network O-D demands using link flow information. Similar idea was proved to be feasible by using the Generalized Gravity model where traffic assignment was conducted by using the proportional assignment principle without link capacity constraints. Later, the Entropy Maximization (EM) approach was applied to solve the network flow estimation problem where prior O-D demand information and link flow measurements are assumed to be known/observable to infer the most likely estimated network O-D demand matrices (Willumsen, 1978). van Zuylen and Willumsen (1980) assumed a set of known path flows and adopted the EM and Information Minimization (IM) approaches to estimate the most possible O-D matrices which are consistent with the prior O-D demand information and observed link/path flow measurements. Similarly, Nihan and Davis (1989) applied the Maximum Likelihood (ML) method to solve the network O-D demand matrices estimation problem where the proposed model assumes that full information on link flows is available, and input/output flow is conserved for an efficient estimate on network O-D demand matrices. However, since full information on link flow measurements is difficult and/or costly to obtain in practice, they further applied the Expectation Maximization (EM) method to infer the maximum likely estimate on a set of network O-D demands.

Another development direction on the methodological aspect is the application of the Least Squares (LS) based approach. Cascetta (1984) proposed both deterministic and stochastic Generalized Least Squares (GLS) models to solve the network O-D demand estimation problem. The problem was formulated as a multiple goals mathematical program and different weights were imposed on both goals: 1) the difference between observed and estimated link flows, and 2) prior and estimated O-D demands. Later, Bell (1991) employed the GLS model proposed by Cascetta (1984) by incorporating the non-negativity constraint on the decision variable, and proposed a constrained GLS model (CGLS) for the problem. Yang (1995) proposed a bi-level model where the upper level model is adopted from the CGLS model developed by Bell (1991) and the lower level model formulates traffic flow distributions by the user equilibrium (UE) traffic assignment principle. The bi-level model was solved by two heuristic solution algorithms, whose solutions are obtained in a time efficient and quick convergence property. Lundgren and Peterson (2008) proposed a heuristic based nonlinear bi-level mathematical program to model the network O-D demand estimation problem. The problem was reformulated as a single-level mathematical program where the objective function is composed of both link flow and O-D flow covariance variables. Similarly, Nie and Zhang (2010) relaxed the UE condition in the lower level model in a bi-level network O-D demand estimation model, which became a single-level mathematical program.

Additionally, Path Flow Estimation (PFE) based approach is applied to investigate the network O-D demand estimation problem. For instance, Sherali *et al*. (1994) developed a Linear PEF (LPFE) model where link flow measurements based on a non-proportional, UE traffic assignment principle are used to infer unknown network O-D demands. Nie and Lee (2002) proposed a degenerated PFE algorithm to solve the network O-D demand estimation problem using those link flows followed an UE traffic assignment principle. Later, Nie *et al*. (2005) incorporated the degenerated PFE approach into a GLS model framework (GLS-PFE), and the problem was solved by using the ACPFE (algorithms for constrained PFE) algorithm where the non-inverse property of a linear system and non-negativity characteristics of the decision variable are guaranteed. Similar PEF based models were proposed to solve the network O-D demand matrix estimation problems (Chootinan *et al*., 2005; Chen *et al*., 2009).

State-of-the-art methods for the estimation of time-dependent network O-D matrices are essentially similar to those for the static network O-D flow estimation problem. Cascetta *et al*. (1993) applied a LS based model to infer time-dependent network O-D demands using the link flow measurements collected at each time interval. Chen and Hsueh (1998) proposed a spatio-temporal network to depict the inflow rate and temporal issue of flow dispersion in a given link. Sherali and Park (2001) constructed a GLS based model for the time-dependent network O-D demand estimation problem. In solving this problem, column generation method was adopted to find the order of dynamic shortest paths, and time-dependent path and/or O-D flows were estimated by an optimal parametric approach in light of a set of time-varying link flow measurements. Zhou and List (2010) proposed an integrated model framework for the optimal sensor location problem under the goal of providing a desirable set of network O-D demand estimates. Nie and Zhang (2008) adopted a variation inequality (VI) approach to solve the dynamic network O-D demand matrix estimation problem. The VI-based method solves the dynamic network flow estimation problem with the shortest path flow information given by the column generation algorithm. The proposed VI model avoids model complexity and a complicated solution procedure generally confronted by the bi-level based models. However, the VI-based model assumes an UE traffic assignment on the observed link flow distribution, which might not be existed, especially in a large scale, dynamic traffic flow estimation problem.

In addition, follow the static model framework proposed by Wang and Chang (2013), the CIM algorithm is employed to solve the dynamic O-D demand estimates in a system of linear equations. Wang and Wu (2013) applied the CIM solution method to time-dependent network O-D demand estimation problem, the developed model is able to solve the network science problem under different flow distributions, whether the link flow is collected under a time-dependent user equilibrium (TDUE) condition or not. The proposed model framework and solution algorithm of Wang and Wu (2013) can reasonably depict the spatio-temporal relationship of dynamic flow evolution without any unreasonable model assumption. But in the concluding remark of Wang and Wu (2013), they also pointed out that the problem about multiple solutions should be solved in future research.

For a moderate general network with hundreds of nodes/links, to solve the dynamic O-D demand estimation problem, one needs to simultaneously consider the temporal issue of traffic flow dispersion and spatial issue of travelers' route choice behavior. This research proposes a spatio-temporal network model adopted from Chen and Hsueh (1998) to describe the flow evolution/distribution for a set of O-D and/or path flow on link flows.

The methodological approaches applied to solve both the static and dynamic network O-D demand estimation problems are briefly classified into: 1) statistical inferring methods, 2) mathematical programming models, 3) advanced filtering techniques. We can compare these methods as follows:

1. The methods about LS-based and PFE-based models are mathematical programming models. Although the VI model is not a mathematical programming model, its sub-problem is also a mathematical programming model.
2. VI-based and PFE-based models assume an UE link traffic flow distribution to guarantee a feasible solution on a network O-D demand matrix estimate. Such an UE assumption on link traffic flow distribution might not exist for a large scale network and/or significant traffic flow variations in a short period of time.
3. In addition, LS based statistical inferring approach does not guarantee that it is a least square between real O-D demands and historical O-D demands, and usually confront with the problem if a feasible solution is existed.
4. The CIM algorithm is not a mathematical programming and different from LS-based, VI-based and PFE-based models. It is a system of linear equations and it uses conditional inverse to solve the O-D demand estimation problem.
5. The CIM algorithm is able to solve the network science problem under different flow distributions, whether the link flow is collected under a time-dependent user equilibrium (TDUE) condition or not.
6. No suitable method is developed to deal with the issue of multiple solutions problem for the O-D demand estimation problem.

In this study, the issue of multiple solutions generally found in the network O-D demand estimation problem is discussed and solved by the path flow proportion method where time-dependent path flow proportions between each O-D pair are assumed to be known. Details of the model framework and solution algorithms will be described in the later sections.

3. THE MODELS

For a general network described by its link-path incidence matrix, the relationships between a set of O-D demands and path/link flows can be formulated as shown in Eqs. (1) and (2), where $\mathbf{q}$ is a vector of O-D flows, $\mathbf{h}$ is a vector of path flows, $\mathbf{f}$ is a vector of link flow, $\mathbf{\Lambda}_1$ is the O-D-path incidence matrix, and $\mathbf{\Lambda}_2$ the link-path incidence matrix.

$$\mathbf{q} = \mathbf{\Lambda}_1\mathbf{h} \tag{1}$$

$$\mathbf{f} = \mathbf{\Lambda}_2\mathbf{h} \tag{2}$$

By combining both Eqs. (1) and (2) and adopting appropriate matrix inverse lemma, one could obtain network O-D demand estimates, such as the derivation shown in Eqs. (3) and (4).

$$\mathbf{f} = \mathbf{\Lambda}_2\mathbf{h} \Rightarrow \mathbf{h} = \mathbf{\Lambda}_2^{-1}\mathbf{f} \tag{3}$$

$$\mathbf{q} = \mathbf{\Lambda}_1\mathbf{h} = \mathbf{\Lambda}_1\mathbf{\Lambda}_2^{-1}\mathbf{f} \tag{4}$$

The system of linear equations shown in Eq. (3) or Eq. (4) could be solved as long as the incidence matrix is a square matrix. However, most of the time the number of used paths is greater than that of the links; the incidence matrix is usually non-squared, or a square matrix but singular. For example, the inverse matrix of $\mathbf{\Lambda}_2$ might not be available since $\mathbf{\Lambda}_2$ is generally non-squared or singular. Graybill (1983) provided a Conditional Inverse Matrix (CIM) method to deal with the non-full rank matrix inverse problem.

Wang and Chang (2013) formulated the static network O-D demand estimation problem as a system of linear equations shown in Eqs. (1) and (2), and solved this problem by the column generation method and the CIM approach. Wang and Wu (2013) also applied the formulated the Eqs. (1) and (2) and the CIM solution method in time-dependent network O-D demand estimation problem. In the present study, we tackle the time-dependent O-D demand estimation problem by a similar modeling framework of Wang and Wu (2013) and discussed how to approach the situation of multiple solutions. According the approach of Wang and Wu (2013) for time-dependent O-D estimated problem, the time-space network should be first constructed to represent the spatio-temporal relationships between a set of unknown O-D/path flows and observed link flows. For time-dependent network problems, the link flow f_a in each time interval also should be expressed as three components which are inflow rate $u_a(t)$, exit flow rate $v_a(t)$ and number of vehicle $x_a(t)$. The relationship between $u_a(t)$, $v_a(t)$ and $x_a(t)$ can be expressed as follows:

$$u_a(t) = v_a(t + \tau_a(t)) \qquad \forall a,t$$

$$x_a(t) = x_a(t-1) + u_a(t-1) - v_a(t-1)$$

Wang and Wu (2013) solved this problem by using the column generation method and CIM method. In this study, we focus on the situations of multiple solutions discussion and develop the path flow proportion method to solve the problem. Details of these models are described below.

3.1 Time-Space Network

A time-space network describes the spatio-temporal relationships among a set of O-D/path flows traversing different links in a physical network over a given time period. We follow the time-space network structure of Chen (1999) to construct the flow propagation phenomena. When the estimated travel time of a link a, $\tau_a(t)$ is temporarily fixed, the relationship among the inflow, exit flow, and number of vehicles on this specific link can be determined through a link flow propagation constraint. A simple physical network example which includes two links and three nodes is shown in Figure 1(a), and the corresponding time-space network with the estimated link travel time information can be drawn and shown in Figure 1(b).

The time-space network essentially contains two dimensions: the horizontal axis denotes spatial distance, and the vertical axis represents time interval. At each time interval, a static network is reproduced. In addition, the dummy links (with zero travel time) between the time-dependent destinations $s(t)$ and the time-independent destination s are also artificially created. Note that the static links are not directly used, instead a new set of links called *time-space links* has been created. A time-space link is created by connecting a tail node at an earlier interval with a head node at a later interval, whereas the estimated link travel time for inflows entering the time-space link a during interval t is represented by $\tau_a(t)$. In other words, for any time-space link a, the inflows $u_a(t)$ during interval t must be equal to the corresponding exit flow $v_a(t+\tau_a(t))$ during time interval $(t+\tau_a(t))$. For example, as shown in Figure 1(b), $v_a(5)$ is the sum of $u_a(2)$ and $u_a(3)$. Furthermore, the slope of a specific time-space link a denotes the inverse of the vehicular traveling speed on that specific link. The steeper the slope, the lower the vehicular traveling speed, and vice versa. As to the number of vehicles remaining on link a at the beginning of time interval t, it can be computed by summing up all of the inflows, $u_a(\bullet)$, passing through link a (except for a tail node) during interval t. For example, also shown in Figure 1(b), $x_a(5)$ is equal to $u_a(2)+u_a(3)+u_a(4)$ whereas $u_a(5)$ is not included, and the number of vehicles on link a at the beginning of interval 7 is equal to $u_a(4)+u_a(5)+u_a(6)$.

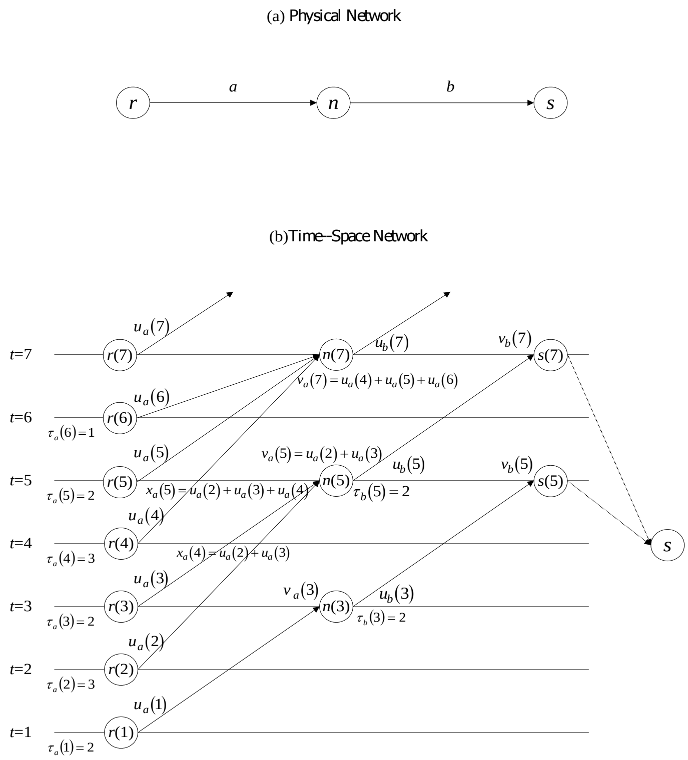

Figure 1 Time-Space Network (Source: Chen, 1999).

3.2 Column Generation Method

The column generation method is applied to create the time-dependent O-D pair-path incidence matrix and time-dependent link-path incidence matrix. The time-dependent O-D pair-path incidence matrix is a zero-one matrix and it represents the spaito-temporal trajectories between a set of O-D pairs and their used paths. Similarly, the time-dependent link-path incidence matrix is a zero-one matrix and it represents the trajectories of paths traversing different links. For example, Eq. (5) expresses a time-dependent O-D pair-path incidence matrix which formulates the spatial relationships between two O-D pairs and four paths where the first two paths are associated with O-D pair 1, and the last two paths are corresponding to O-D pair 2. In addition, Eq. (6) is a time-dependent link-path incidence matrix, which formulates the spatial relationships between four paths and four links where path 1 traverses through links 1 and 4, path 2 traverses through links 2 and 3, path 3 traverses through link 3 and 4, and path 4 traverses links 1 through 3.

$$\mathbf{\Lambda}_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{5}$$

$$\mathbf{\Lambda}_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \tag{6}$$

3.3 Conditional Inverse Matrix Approach and Path Flow Proportion Method

The conditional inverse of a matrix can be described by the following definitions and theorems (Graybill, 1983):

Definition 1: Let $\mathbf{A}$ be a $m \times n$ matrix. $\mathbf{A}^C$ is defined as a conditional inverse matrix of $\mathbf{A}$, if and only if $\mathbf{A}^C$ satisfies $\mathbf{A}\mathbf{A}^C\mathbf{A} = \mathbf{A}$.

Definition 2: A $n \times n$ matrix $\mathbf{H}$ is defined as an upper Hermite form if and only if it satisfies the following four conditions.

1) $\mathbf{H}$ is upper triangular.
2) The diagonal elements of the matrix are only 0 and 1.
3) If a row has a 0 element on the diagonal, then every element in that row is 0.
4) If a row has a 1 element on the diagonal, then every off-diagonal element is 0 in that column in which the 1 appears.

Theorem 1: A conditional inverse exists for each matrix, but it may not be unique.

Theorem 2: If $\mathbf{A}$ is a $m \times n$ matrix, a conditional inverse of $\mathbf{A}$ is a $n \times m$ matrix.

Theorem 3: if $\mathbf{H}$ is in Hermite form, then $\mathbf{H} = \mathbf{H}^2$.

Theorem 4: For any $n \times n$ matrix $\mathbf{A}$, there exists a nonsingular matrix $\mathbf{B}$ such that $\mathbf{BA} = \mathbf{H}$, where $\mathbf{H}$ is in Hermite form.

Theorem 5: Let $\mathbf{A}$ be a $n \times n$ matrix. Let $\mathbf{B}$ be a nonsingular matrix such that $\mathbf{BA} = \mathbf{H}$, where $\mathbf{H}$ is in Hermite form. Then $\mathbf{B}$ is a conditional inverse of $\mathbf{A}$.

Corollary 5: Let A be a $m \times n$ matrix with $n > m$, and let $\mathbf{A}_0 = [\mathbf{A}, \mathbf{0}]$ where 0 is the $m \times (m-n)$ 0 matrix. Let $\mathbf{B}_0$ be a nonsingular matrix such that $\mathbf{B}_0\mathbf{A}_0 = \mathbf{H}$, where $\mathbf{H}$ is in Hermite form. Let $\mathbf{B}_0$ be partitioned as $\mathbf{B}_0 = [\mathbf{B}, \mathbf{B}_1]$ where $\mathbf{B}$ is a $n \times m$ matrix. Then $\mathbf{B}$ is a conditional inverse of $\mathbf{A}$. A similar corollary can be obtained for the situation that $n < m$.

Theorem 6: For any conditional inverse $\mathbf{A}^C$ of a $m \times n$ matrix $\mathbf{A}$, the matrices $\mathbf{A}^C\mathbf{A}$ and $\mathbf{A}\mathbf{A}^C$ are each idempotent.

Theorem 7: If A is a $n \times n$ nonsingular matrix, then the system $\mathbf{Ax} = \mathbf{g}$ has a unique solution.

Theorem 8: Let $\mathbf{A}$ be a $m \times n$ matrix, and let $\mathbf{A}^C$ be any conditional inverse of $\mathbf{A}$, suppose a solution exist for the system $\mathbf{Ax} = \mathbf{g}$. For each $n \times 1$ vector $\mathbf{l}$, the vector $\mathbf{x}_0$ is a solution, where $\mathbf{x}_0 = \mathbf{A}^C\mathbf{g} + (\mathbf{I} - \mathbf{A}^C\mathbf{A})\mathbf{l}$.

Consider the time-dependent traffic flows on a given network, two sets of system equations are respectively formulated as Eqs. (7) and (8) for the spatio-temporal relationships between a vector of O-D and path flows at each departure time interval, and a set of path and link flows at various time intervals.

$$\mathbf{q}(\mathbf{k}) = \mathbf{\Lambda}_1 \mathbf{h}(\mathbf{k}) \tag{7}$$
$$\mathbf{u}(\mathbf{t}) = \mathbf{\Lambda}_2 \mathbf{h}(\mathbf{k}) \tag{8}$$

where $\mathbf{q}(\mathbf{k})$ is a vector of O-D flows departed their origins at time interval *k*. $\mathbf{\Lambda}_1$ is the O-D pair/path incidence matrix during departure time interval *k*. $\mathbf{h}(\mathbf{k})$ is a vector of path flows during the departure time intervals *k*. $\mathbf{u}(\mathbf{t})$ is a vector of inflows for link *a* during time interval *t*. $\mathbf{\Lambda}_2$ is the link-path incidence matrix during each time interval *t*.

According to the formulations shown in Eqs. (7) and (8), and by adopting Theorems 4 and 5, let $\mathbf{B}$ be a nonsingular matrix such that $\mathbf{BA} = \mathbf{H}$ where $\mathbf{H}$ is in Hermite form. $\mathbf{B}$ is a conditional inverse matrix of $\mathbf{A}$ and denoted by $\mathbf{B} = \mathbf{A}^C$. We can obtain both path and O-D flow estimates via the CIM as follows.

$$\mathbf{h}(\mathbf{k}) = \mathbf{\Lambda}_2^C \mathbf{u}(\mathbf{t}) \tag{9}$$
$$\mathbf{q}(\mathbf{k}) = \mathbf{\Lambda}_1 \mathbf{h}(\mathbf{k}) = \mathbf{\Lambda}_1 \mathbf{\Lambda}_2^C \mathbf{u}(\mathbf{t}) \tag{10}$$

Proof 1: Given $\mathbf{\Lambda}_2 \mathbf{h}(\mathbf{k}) = \mathbf{u}(\mathbf{t})$, and $\mathbf{\Lambda}_2 \neq \mathbf{0}$, multiply the left hand side by $\mathbf{\Lambda}_2 \mathbf{\Lambda}_2^C$, we have:

$$\mathbf{\Lambda}_2 \mathbf{\Lambda}_2^C \mathbf{h}(\mathbf{k}) = \mathbf{\Lambda}_2 \mathbf{\Lambda}_2^C \mathbf{u}(\mathbf{t}). \tag{11}$$

According to Definition 1, the left hand side of Eq. (11) can be derived as follows: $\mathbf{\Lambda}_2 \mathbf{\Lambda}_2^C \mathbf{\Lambda}_2 \mathbf{h}(\mathbf{k}) = \mathbf{\Lambda}_2 \mathbf{h}(\mathbf{k}) = \mathbf{u}(\mathbf{t})$; therefore, $\mathbf{\Lambda}_2 \mathbf{\Lambda}_2^C \mathbf{u}(\mathbf{t}) = \mathbf{u}(\mathbf{t})$.
Next, assume $\mathbf{\Lambda}_2 \mathbf{\Lambda}_2^C \mathbf{u}(\mathbf{t}) = \mathbf{u}(\mathbf{t})$, and let $\mathbf{h}(\mathbf{k}) = \mathbf{\Lambda}_2^C \mathbf{u}(\mathbf{t})$.
If we substitute this value for $\mathbf{h}(\mathbf{k})$ into the system $\mathbf{\Lambda}_2 \mathbf{h}(\mathbf{k}) = \mathbf{u}(\mathbf{t})$, we then have: $\mathbf{\Lambda}_2 \mathbf{\Lambda}_2^C \mathbf{u}(\mathbf{t}) = \mathbf{u}(\mathbf{t})$.
Hence, $\mathbf{h}(\mathbf{k}) = \mathbf{\Lambda}_2^C \mathbf{u}(\mathbf{t})$ is a solution of Eq. (10).
Therefore, $\mathbf{q}(\mathbf{k}) = \mathbf{\Lambda}_1 \mathbf{h}(\mathbf{k}) = \mathbf{\Lambda}_1 \mathbf{\Lambda}_2^C \mathbf{u}(\mathbf{t})$, and this completes the proof.

Let matrix $\mathbf{H}$ satisfy the four Hermite form conditions of Definition 2, an augmented matrix $\left[\mathbf{\Lambda}_2 \middle| \mathbf{I}\right] = \left[\mathbf{H} \middle| \mathbf{\Lambda}_2^C\right]$ can be derived through the Gaussian-Jordan elimination method. Thereby, the conditional inverse $\mathbf{\Lambda}_2^C$ can be accordingly derived. Note that the time-dependent link-path incidence matrix $\mathbf{\Lambda}_2$ does not guarantee a $n \times n$ square matrix. It can be filled by a **0** vector to become a square matrix in implementing the CIM matrix inverse procedure.

From Theorem 1 and the previous proof 1, we can find that the result given by conditional inverse matrix method may not be unique. That is, $\mathbf{h}(\mathbf{k})$ is one of the solutions of $\mathbf{\Lambda}_2^C \mathbf{u}(\mathbf{t})$, but it does not guarantee the uniqueness. Gentili and Mirchandani (2005) and Castillo *et al.* (2008) pointed out that when the number of paths and O-D pairs are large than the number of links in a network, there are more than one solution for estimating O-D demands from link traffic flows. In order to estimate the exact O-D demands from link traffic flows, it is necessary to analyze the link/path incidence matrix $\mathbf{\Lambda}_2$.

To analyze a linear system, $\mathbf{Ax}=\mathbf{b}$, let $\mathbf{A}$ be a $m \times n$ matrix. It has a unique solution for row rank **A** = *n*. If rank **A** = *r* < *n*, there are multiple solutions for the linear system, $\mathbf{Ax}=\mathbf{b}$. For a $m \times n$ matrix $\mathbf{A}$, the number of row ranks can be derived by its row reduced echelon form (RREF) $\mathbf{R}$. The row reduced echelon form $\mathbf{R}$ should satisfy the following properties (Friedberg *et al.*, 2003):

1) All zero rows, if there are any, appear at the bottom of the matrix.
2) The first nonzero entry from the left of a nonzero row is a 1. The entry is called leading one of the row.
3) For each nonzero row, the leading one appears to the right and below any leading one's in preceding rows.

Compare the Hermite form $\mathbf{H}$ and row reduced echelon form $\mathbf{R}$, we can find that the properties of the Hermite form $\mathbf{H}$ is similar to row reduced echelon form $\mathbf{R}$. Therefore, there are two scenarios can be obtained when applying the conditional inverse method to estimate network O-D demands, depicted below.

I. The Hermite form $\mathbf{H}$ is an identity matrix $\mathbf{I}$, then rank($\mathbf{\Lambda}_2$)= *r* = column *n*. For the linear system $\mathbf{\Lambda}_2\mathbf{h}_{\mathbf{p}}^{\mathbf{rs}}(\mathbf{k})=\mathbf{u}_{\mathbf{a}}(\mathbf{t})$, there is a unique solution $\mathbf{h}(\mathbf{k})=\mathbf{\Lambda}_2^{C}\mathbf{u}(\mathbf{t})$.

II. The Hermite form $\mathbf{H}$ is not an identity matrix $\mathbf{I}$, then rank($\mathbf{\Lambda}_2$)= *r* < column *n*. There are multiple solutions $\mathbf{h}_{\mathbf{p}}^{\mathbf{rs}}(\mathbf{k})$ for the linear system $\mathbf{\Lambda}_2\mathbf{h}_{\mathbf{p}}^{\mathbf{rs}}(\mathbf{k})=\mathbf{u}_{\mathbf{a}}(\mathbf{t})$.

Under scenario I, the network O-D demand estimation problem can be directly solved by the conditional inverse approach. However, the problem cannot be uniquely solved by the conditional inverse method under scenario II. In order to have a unique solution, it is necessary to incorporate more information into the linear system $\mathbf{\Lambda}_2\mathbf{h}_{\mathbf{p}}^{\mathbf{rs}}(\mathbf{k})=\mathbf{u}_{\mathbf{a}}(\mathbf{t})$. Here, we assume the proportions of the used paths between each O-D pair in each time interval are known, and develop the "path flow proportion method".

We can build a path flow proportion/O-D incidence matrix, $\mathbf{P}$, such as Table 1 to represent the path flow proportions between each O-D pair in each time interval. For instance, in Table 1, paths 1 and 2 shown on the respective rows express the proportions are 0.7 and 0.3 between O-D pair 1 at time interval 1; and paths 3 and 4 shown on the respective rows express the proportions are 0.4 and 0.6 between O-D pair 2 at time interval 2.

Table 1 Path Flow Proportion/O-D Incidence Matrix

Path No.	Time interval	O-D 1		O-D 2	
		path 1	path 2	path 3	path 4
path 1	*k*=1	0.7	0	0	0
path 2	*k*=1	0.3	0	0	0
path 3	*k*=2	0	0	0.4	0
path 4	*k*=2	0	0	0.6	0

When the link/path incidence matrix $\mathbf{\Lambda}_2$ multiplies the path flow proportion/O-D incidence matrix $\mathbf{P}$, the link proportion/O-D incidence matrix, $\mathbf{\Lambda}_3$ can be obtained. In $\mathbf{\Lambda}_3$, paths 1 and 2 can be replaced by O-D pair 1, and paths 3 and 4 can be replaced by O-D pair 2. The relationship between link flows and O-D flows can thus be expressed as Eqs. (12) and (13):

$$\mathbf{\Lambda_2 P}=\mathbf{\Lambda_3} \tag{12}$$

$$\mathbf{\Lambda}_3\mathbf{q}(\mathbf{k})=\mathbf{u}(\mathbf{t}) \tag{13}$$

In the equation (13), the assumption of path proportion is known. As a result, the path variables can be replaced by O-D pair variables. It will reduce the number of column in $\mathbf{\Lambda}_3$. If the number of links is large than the number of O-D pairs in a given network, then rank($\mathbf{\Lambda}_3$)= r = column n. For the linear system $\mathbf{\Lambda}_3\mathbf{q}(\mathbf{k})=\mathbf{u}(\mathbf{t})$, a unique solution can be obtained by conditional inverse and shown in equation (14).

$$\mathbf{q}(\mathbf{k})=\mathbf{\Lambda}_3^C\mathbf{u}(\mathbf{t}) \tag{14}$$

The assumption of the path flow proportions are known is a strong assumption for the time-dependent O-D pairs estimation problem. But if we want to obtain a more realistic time-dependent O-D pairs estimating result, it is necessary to add more information into the model especially O-D pair estimation is a multiple solutions problem. In theory, if the path variables can be replaced by O-D pair variables through the transfer of path flow proportions, adopting link flows to estimating time-dependent O-D matrix could be guaranteed. On the other hand, if most of road users are frequent users in an urban network, it is possible to obtain the information of path flow proportions between each time-dependent O-D pair through sampling survey. The assumption on the known path flow proportions also can be relaxed by obtaining users' route choice probabilities using some advanced sensor technologies (e.g., AVI or license plate recognition technologies) or sensor location algorithms (e.g. Gentili and Mirchandani, 2005).

4. THE SOLUTION ALGORITHM

In the present study, the spatio-temporal relationship between a set of unknown O-D/path flows and observed link flows is constructed by a time-space network. Under scenarios I and II depicted before, the time-dependent O-D matrix estimation problems can be respectively solved by the path flow proportion method under the conditional inverse matrix approach structure, and the corresponding solution steps are shown below.

Step 1: Construct a time-space network for a given physical network. Obtain the link travel times during each time interval and use the observed link travel times to transfer the physical network into a time-space network.

Step 2: Build the path proportion/O-D incidence matrix $\mathbf{P}$ using the known time-space path proportions between each O-D pair at each time interval, and obtain the corresponding time-dependent link-path incidence matrix $\mathbf{\Lambda}_2$. Use Eq. (12) to obtain the time-dependent link proportion/O-D incidence matrix $\mathbf{\Lambda}_3$

Step 3: Compute the conditional inverse matrix of $\mathbf{\Lambda}_3$ by establishing an augmented matrix $\left[\mathbf{\Lambda}_3 \middle| \mathbf{I}\right]=\left[\mathbf{H} \middle| \mathbf{\Lambda}_3^C\right]$ and solve $\mathbf{\Lambda}_3^{\mathbf{C}}$ through the Gaussian-Jordan elimination method. Let matrix $\mathbf{H}$ satisfy the four Hermite form conditions of Definition 2, then the conditional inverse $\mathbf{\Lambda}_3^{\mathbf{C}}$ can be derived.

Step 4: Compute the time-dependent O-D demands using Eq. (14).

5. NUMERICAL ANALYSIS AND DISCUSSION

Based on the model framework and solution procedure described in the previous sections, the present section conducts various numerical analyses to demonstrate the feasibility of the proposed model framework in the estimation of time-dependent O-D matrices. The network used in the numerical analysis (as shown in Figure 2) is adopted from Yang (1995), where nodes 1 and 2 are origins, and nodes 3 and 4 are destinations. For model evaluation purpose a vector of hypothetical true time-varying O-D demands is shown in Table 2. There are totally four time-dependent O-D demands leaving their origins at various time intervals. In the beginning, it is assumed that the link travel time in each time interval is known, and the O-D pairs and departure time intervals for each O-D pair are given in advance. Besides, two test scenarios of different network flow pattern assumptions are designed for evaluating model applicability purposes.

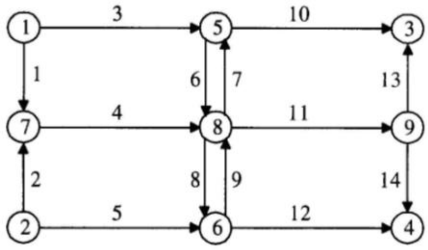

Figure 2 Yang's Network (Source: Yang, 1995).

Table 2 Hypothetical True Time-dependent O-D Demands

No.	Departure time	O-D pair	Demand
1	k=1	1-3	200
2	k=2	1-4	150
3	k=2	2-3	140
4	k=1	2-4	185

Scenario 1: The time-dependent link flow patterns without the DUO Assumption

In here, suppose the path flow proportions between each O-D pair at each time interval are known and shown in Table 3. The observed network flows in each time interval are shown in Table 4 and without the DUO assumption. We will estimate the O-D trips matrix through the processes of path flow proportion method.

Table 3 Time-dependent Path Flow Proportions between Each O-D Pair

No.	Departure time interval (k)	Path	Path flow proportion	No.	Departure time interval (k)	Path	Path flow proportion
1	1	1-7-8-9-3	0.4	9	2	2-6-8-5-3	0.1
2	1	1-5-3	0.2	10	2	2-7-8-9-3	0.1
3	1	1-7-8-5-3	0.1	11	2	2-6-8-9-3	0.3
4	1	1-5-8-9-3	0.3	12	2	2-7-8-5-3	0.5
5	2	1-5-8-6-4	0.2	13	1	2-7-8-9-4	0.2
6	2	1-7-8-6-4	0.3	14	1	2-6-4	0.4
7	2	1-7-8-9-4	0.2	15	1	2-7-8-6-4	0.2
8	2	1-5-8-9-4	0.3	16	1	2-7-8-9-4	0.2

Table 4 Observed Network Flows and Link Travel Times

Link	Entering time interval	In flow	Exit flow	No. of vehicles	Link travel time	Exiting time interval
	1	100	0	0.00	1.06	2
1→5	2	75	100	100.00	1.00	3
	3	0	75	75.00	1.00	4

Link	Entering time interval	In flow	Exit flow	No. of vehicles	Link travel time	Exiting time interval
1→7	1	100	0	0.00	2.00	3
	2	75	0	100.00	2.00	4
	3	0	100	175.00	2.00	5
	4	0	75	75.00	2.00	6
2→6	1	74	0	0.00	3.01	4
	2	56	0	74.00	3.00	5
	3	0	0	130.00	3.00	6
	4	0	74	130.00	3.00	7
	5	0	56	56.00	3.0	8
2→7	1	111	0	0.00	1.00	2
	2	84	111	111.00	1.02	3
	3	0	84	84.00	1.00	4
5→3	2	40	0	0.00	1.15	3
	3	0	40	40.00	1.00	4
	4	0	0	0.00	1.00	5
	5	0	0	0.00	1.00	6
	6	0	0	0.00	1.00	7
	7	90	0	0.00	1.00	8
	8	0	90	90.00	1.00	9
	9	0	0	0.00	1.00	10
	10	14	0	0.00	1.00	11
	11	0	14	14.00	1.00	12
5→8	2	60	0	0.00	2.00	4
	3	75	0	60.00	2.00	5
	4	0	60	135.00	2.00	6
	5	0	75	75.00	2.00	7
6→4	4	111	0	0.00	1.02	5
	5	0	111	111.00	1.00	6
	6	75	0	0.00	1.01	7
	7	0	75	75.00	1.00	8
6→8	5	56	0	0.00	2.00	7
	6	0	0	56.00	2.00	8
	7	0	56	56.00	2.00	9
7→8	2	111	0	0.00	1.00	3
	3	184	111	111.00	1.01	4
	4	75	184	184.00	1.00	5
	5	0	75	75.00	1.00	6
8→5	4	90	0	0.00	3.00	7
	5	0	0	90.00	3.00	8
	6	0	0	90.00	3.00	9
	7	14	90	90.00	3.00	10
	8	0	0	14.00	3.00	11
	9	0	0	14.00	3.00	12
	10	0	14	14.00	3.00	13
8→6	3	37	0	0.00	1.00	4
	4	0	37	37.00	1.00	5
	5	75	0	0.00	1.01	6
	6	0	75	75.00	1.00	7
8→9	3	74	0	0.00	2.00	5
	4	154	0	74.00	2.07	6
	5	75	74	228.00	2.00	7
	6	0	154	229.00	2.00	8
	7	42	75	75.00	2.00	9
	8	0	0	42.00	2.00	10
	9	0	42	42.00	2.00	11
9→3	6	154	0	0.00	1.01	7

Link	Entering time interval	In flow	Exit flow	No. of vehicles	Link travel time	Exiting time interval
	7	0	154	154.00	1.00	8
	8	0	0	0.00	1.00	9
	9	42	0	0.00	1.00	10
	10	0	42	42.00	1.00	11
9→4	5	74	0	0.00	2.00	7
	6	0	0	74.00	2.00	8
	7	75	74	74.00	2.00	9
	8	0	0	75.00	2.00	10
	9	0	75	75.00	2.00	11

Follow the steps of the path flow proportion method depicted in section 4, the solution procedure is as follows:

Step 1: Construct a time-space network for a given physical network. Obtain the link travel times during each time interval and use the observed link travel times to transfer the physical network into a time-space network.

Step 2: Build the path proportion/O-D incidence matrix $\mathbf{P}$ as shown in Eq. (15) and the $\mathbf{\Lambda}_2$ as Table 5. From equation (12) and delete the 0 column, the time-dependent link proportions/O-D incidence matrix $\mathbf{\Lambda}_3$ can be derived as Table 6.

$$\mathbf{P}=\begin{bmatrix}
0.4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0.2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0.1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0.3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0.2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0.3&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0.2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0.3&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0.1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0.1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0.3&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0.5&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0.2&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0.4&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0.2&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0.2&0&0&0
\end{bmatrix} \tag{15}$$

Table 5 Time-dependent Link-path Incidence Matrix, $\mathbf{\Lambda}_2$

No	Link	Time interval	Path no.															
			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1-5	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
2	1-5	2	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0
3	1-7	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1-7	2	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
5	2-6	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
6	2-6	2	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0
7	2-7	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1
8	2-7	2	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0
9	5-3	2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	5-3	7	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0
11	5-3	10	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
12	5-8	2	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
13	5-8	3	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0
14	6-4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
15	6-4	6	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
16	6-8	5	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0
17	7-8	2	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1

18	7-8	3	1	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0
19	7-8	4	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
20	8-5	4	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0
21	8-5	7	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
22	8-6	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
23	8-6	5	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
24	8-9	3	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
25	8-9	4	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0
26	8-9	5	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
27	8-9	7	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
28	9-3	6	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0
29	9-3	9	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
30	9-4	5	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
31	9-4	7	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0

Table 6 Time-dependent Link Proportion/O-D Incidence Matrix, $\mathbf{\Lambda_3}$

No.	Link	Time interval	O-D pair			
			1	2	3	4
1	1-5	1	0.5	0	0	0
2	1-5	2	0	0.5	0	0
3	1-7	1	0.5	0	0	0
4	1-7	2	0	0.5	0	0
5	2-6	1	0	0	0	0.4
6	2-6	2	0	0	0.4	0
7	2-7	1	0	0	0	0.6
8	2-7	2	0	0	0.6	0
9	5-3	2	0.2	0	0	0
10	5-3	7	0.1	0	0.5	0
11	5-3	10	0	0	0.1	0
12	5-8	2	0.3	0	0	0
13	5-8	3	0	0.5	0	0
14	6-4	4	0	0	0	0.6
15	6-4	6	0	0.5	0	0
16	6-8	5	0	0	0.4	0
17	7-8	2	0	0	0	0.6
18	7-8	3	0.5	0	0.6	0
19	7-8	4	0	0.5	0	0
20	8-5	4	0.1	0	0.5	0
21	8-5	7	0	0	0.1	0
22	8-6	3	0	0	0	0.2
23	8-6	5	0	0.5	0	0
24	8-9	3	0	0	0	0.4
25	8-9	4	0.7	0	0.1	0
26	8-9	5	0	0.5	0	0
27	8-9	7	0	0	0.3	0
28	9-3	6	0.7	0	0.1	0
29	9-3	9	0	0	0.3	0
30	9-4	5	0	0	0	0.4
31	9-4	7	0	0.5	0	0

Step 3: Compute the conditional inverse matrix of $\mathbf{\Lambda_3}$ by establishing an augmented matrix $\left[\mathbf{\Lambda}_3 \middle| \mathbf{I}\right] = \left[\mathbf{H} \middle| \mathbf{\Lambda}_3^C\right]$ and solve $\mathbf{\Lambda_3^C}$ through the Gaussian-Jordan elimination method. Let matrix $\mathbf{H}$ satisfy the four Hermite form conditions of Definition 2, then the conditional inverse $\mathbf{\Lambda_3^C}$ can be derived as follows Eq. (16):

$$\mathbf{\Lambda}_3^{C}=\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0.5 & 0 & 0 & 2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0.5 & 2.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (16)$$

Step 4: Compute the time-dependent O-D demands using Eq. (14). The estimated time-dependent O-D demands are shown in Table 7.

Table 7 Time-dependent O-D Demand Estimates

No.	Departure time interval (k)	O-D Pair	True O-D demand	Estimated O-D demand
1	1	1-3	200	200
2	2	1-4	150	150
3	2	2-3	140	140
4	1	2-4	185	185

The numerical analysis result shows that the decision variables can be reduced by the path flow proportion method, and the time-dependent O-D demands can be accurately estimated under the CIM approach framework. For a situation where the Hermite form $\mathbf{H}$ is not an identity matrix $\mathbf{I}$, and rank($\mathbf{\Lambda}_2$)= r < column n, the unique solution can be still accurately obtained by adopting the path flow proportion method.

Scenario 2: The time-dependent link flow patterns with the DUO Assumption

Based on the above experimental setup and assumptions, the observed inflows, exit flows and number of vehicles during different time intervals in the network are obtained under the dynamic user optimal (DUO) condition (Chen, 1999). The DUO is defined as: *for each O-D pair, the actual route travel times experienced by travelers departing at the same interval are equal and the minimal; no traveler could be better off by unilaterally changing his/her route*. In other words, the actual route travel times of any unused route for each O-D pair are greater than or equal to the minimal actual route travel time." Also note that the time duration of a time interval could be five to 30 minutes depending on the purposes of traffic control and/or monitoring. The link inflows in each time interval results in terms of the network flow distribution obey the DUO principle are shown in Table 8. The path flow proportions between each O-D pair at each time interval are shown in Table 9.

To evaluate whether the time-dependent O-D demands can be inferred by a set of observed time-dependent link flows pattern with DUO conditions. We also follow the steps of the path flow proportion method depicted in section 4 and derive the estimation O-D demands which are shown on table 10.

Table 8 Observed Network Flows and Link Travel Times

Link	Entering time interval	In flow	Exit flow	No. of vehicles	Link travel time	Exiting time interval
1→5	1	147.21	0.00	0.00	3.17	4
	2	79.78	0.00	147.21	3.80	6
	3	0.00	0.00	226.99	6.15	--
	4	0.00	147.21	226.99	6.15	--
	5	0.00	0.00	79.78	1.64	--
	6	0.00	79.78	79.78	1.64	--
1→7	1	52.79	0.00	0.00	1.28	2

	2	70.22	52.79	52.79	1.77	5
	3	0.00	0.00	70.22	1.49	--
	4	0.00	70.22	70.22	1.49	--
2→6	1	143.54	0.00	0.00	3.06	4
	2	76.80	0.00	143.54	3.65	6
	3	0.00	0.00	220.33	5.85	--
	4	0.00	143.54	220.33	5.85	--
	5	0.00	0.00	76.80	1.59	--
	6	0.00	76.80	76.80	1.59	--
2→7	1	41.46	0.00	0.00	1.17	2
	2	63.20	41.46	41.46	1.57	4
	3	0.00	0.00	63.20	1.40	--
	4	0.00	63.20	63.20	1.40	--
5→3	4	147.21	0.00	0.00	3.17	7
	5	0.00	0.00	147.21	3.17	--
	6	0.00	0.00	147.21	3.17	--
	7	0.00	147.21	147.21	3.17	--
	10	87.63	0.00	0.00	1.77	12
	11	0.00	0.00	87.63	1.77	--
	12	0.00	87.63	87.63	1.77	--
5→8	6	79.78	0.00	0.00	1.64	8
	7	0.00	0.00	79.78	1.64	--
	8	0.00	79.78	79.78	1.64	--
6→4	4	143.54	0.00	0.00	3.06	7
	5	0.00	0.00	143.54	3.06	--
	6	0.00	0.00	143.54	3.06	--
	7	0.00	143.54	143.54	3.06	--
	10	90.04	0.00	0.00	1.81	12
	11	0.00	0.00	90.04	1.81	--
	12	0.00	90.04	90.04	1.81	--
6→8	6	76.80	0.00	0.00	1.59	8
	7	0.00	0.00	76.80	1.59	--
	8	0.00	76.80	76.80	1.59	--
7→8	2	94.25	0.00	0.00	1.89	4
	3	0.00	0.00	94.25	1.89	--
	4	133.40	94.25	94.25	3.67	8
	5	0.00	0.00	133.42	2.78	--
	6	0.00	0.00	133.42	2.78	--
	7	0.00	0.00	133.42	2.78	--
	8	0.00	133.42	133.42	2.78	--
8→5	8	87.63	0.00	0.00	1.77	10
	9	0.00	0.00	87.63	1.77	--
	10	0.00	87.63	87.63	1.77	--
8→6	8	90.04	0.00	0.00	1.81	10
	9	0.00	0.00	90.04	1.81	--
	10	0.00	90.04	90.04	1.81	--
8→9	4	94.25	0.00	0.00	1.89	6
	5	0.00	0.00	94.25	1.89	--
	6	0.00	94.25	94.25	1.89	--
	8	112.33	0.00	0.00	2.26	10
	9	0.00	0.00	112.33	2.26	--
	10	0.00	112.33	112.33	2.26	--
9→3	6	52.79	0.00	0.00	1.28	7
	7	0.00	52.79	52.79	1.28	--
	10	52.37	0.00	0.00	1.27	11
	11	0.00	52.37	52.37	1.27	--
9→4	6	41.46	0.00	0.00	1.17	7
	7	0.00	41.46	41.46	1.17	--
	10	59.96	0.00	0.00	1.36	11

11	0.00	59.96	59.96	1.36	--

Table 9 Time-dependent Path Flow Proportions between Each O-D Pair

No.	Departure time interval (k)	Path	Path flow proportion	No.	Departure time interval (k)	Path	Path flow proportion
1	1	1-7-8-9-3	0.264	7	2	2-7-8-9-3	0.342
2	1	1-5-3	0.736	8	2	2-7-8-5-3	0.109
3	2	1-5-8-9-4	0.343	9	2	2-6-8-9-3	0.032
4	2	1-5-8-6-4	0.189	10	2	2-6-8-5-3	0.517
5	2	1-7-8-6-4	0.411	11	1	2-6-4	0.776
6	2	1-7-8-9-4	0.057	12	1	2-7-8-9-4	0.224

Table 10 Time-dependent O-D Demand Estimates

No.	Departure time interval (k)	O-D pair	True O-D demand	Estimated O-D demand
1	1	1-3	200	200
2	2	1-4	150	150
3	2	2-3	140	140
4	1	2-4	185	185

Based on the numerical test results we also can find that the estimated time-dependent O-D demands are exactly the same as true time-dependent O-D demands. The time-dependent O-D demands can be accurately estimated by the proposed model and solution procedure. The numerical analysis result reveals that the time-dependent O-D demand estimates by solving the proposed model under different test scenarios of traffic flow distributions and departure times are highly accurate. The proposed model framework can be applied to an on-line traffic management scheme by providing highway users with desirable routing suggestions based on the potential trip distribution over a period of time under different traffic flow equilibrium principles.

6. CONCLUDING REMARKS

In this study, we develop a link flow-based time-dependent O-D demand estimation method and solution algorithms by adopting the conditional inverse matrix approach and path flow proportion method. In view of the unique capability of the conditional inverse matrix method in solving a non-squared matrix inverse problem, the time-dependent the link proportion/O-D incidence matrix $\mathbf{\Lambda}_3$ which is usually non-squared is effectively obtained. Through the property of conditional inverse matrix approach and path flow proportion method, the conditional inverse matrix of a time-dependent link proportion/O-D incidence matrix can be derived and the time-dependent O-D demands can be accordingly estimated. Based on the numerical test results, the following conclusions are drawn.

1) The model developed and solution algorithm in this research also can further handle various link flow distribution situations, whether the link flow is given by the DUO principle or not. The proposed model framework essentially provides accurate network O-D demand estimates on a time-dependent basis.
2) On the assumption of path flow proportions are known, the unique solution had been guaranteed when the number of links is large than the number of O-D pairs in a given network.
3) Due to the proposed solution method is not an iteration algorithm. It is very simple and with high solution performance.
4) In this study, we focus on the demonstration of the proposed method development and step-by-step procedure. In future research, more numerical experiments using

larger network with real world data are necessary to justify the proposed model framework and adopted solution algorithms.

For the O-D demand estimation problem, multiple solutions might exist. Under such a circumstance, we can solve the network science problem by the "path flow proportion method". Despite the assumption on the known used path flow proportion between each O-D pair at each time interval might be unavailable and/or difficult to obtain in practice, one of the purposes of this research is to illustrate that a unique solution for the time-dependent O-D estimation problem can be theoretically derived, and desirable network O-D demand estimates can be obtained without any further assumption. In the present research, the multiple solutions issue of the time-dependent O-D matrices estimation problem has been discussed. We assume that the proportions of the used paths between each O-D pair in each time interval are known, and develop the path flow proportion method. As a result, the path variables can be replaced by O-D pair variables. It will significantly reduce the number of columns in the link proportion/O-D incidence matrix, $\mathbf{\Lambda}_3$. If the number of links is larger than the number of O-D pairs in a given network, then rank($\mathbf{\Lambda}_3$)= r = column n. For the linear system $\mathbf{\Lambda}_3\mathbf{q}(\mathbf{k})=\mathbf{u}(\mathbf{t})$, a unique solution can be obtained by the conditional inverse $\mathbf{q}(\mathbf{k})=\mathbf{\Lambda}_3^C\mathbf{u}(\mathbf{t})$. The problem of multiple solutions for estimating time-dependent O-D trip matrices problem can be solved by the path flow proportion method. But how to obtain the path flow proportions between each O-D pair should be discussed.

For an urban network, if most of road users are frequent network users, the path flow proportions between each O-D pair could be estimated by sampling survey. The assumption on the known path flow proportions also can be relaxed by obtaining users' route choice probabilities using some advanced sensor technologies (e.g., AVI or license plate recognition technologies). Due to the information of path flow proportions is the most important key element for estimating O-D matrices from link traffic flows, the effect of sampling error rate is a necessary problem to be discussed. The O-D demand estimation accuracy using the proposed path flow proportion method is affected by sampling error rate should be investigated in the near future. For practical applications, to construct a desirable sampling survey mechanism for estimating path flow proportions between each O-D pair in each time interval is also an important study direction for the time-dependent O-D demand estimation problem.

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