$\forall b, o, d, \tau, p \in P(b, o, d, \tau)\}$; $GC^*(r^l) \equiv \{GC^{\tau}_{odp^*}(b, r^l), \forall b, o, d, \tau, p \in P(b, o, d, \tau)\}$ is the multi-class least path generalized cost vector evaluated at a flow pattern r^l.

Proposition 2: $(-Dir^l)$ is the feasible descent searching direction corresponding to the iterate r^l.
Proof: the proof of Proposition 2 is given in the Appendix.

According to Eq.(15), the new iterate r^{l+1} is obtained by updating the current iterate r^l along the direction $-Dir^l$ with a move size ρ^l. Specifically, for each (b, o, d, τ) sub-problem, the proposed multi-class path-swapping descent direction method updates the current path assignment r^l as follows:

$$r^{\tau,l+1}_{odp}(b) = \max\{0,\ r^{\tau,l}_{odp}(b) - \rho^l \times \frac{r^{\tau,l}_{odp}(b) \times [GC^{\tau}_{odp}(b, r^l) - GC^{\tau}_{odp^*}(b, r^l)]}{GC^{\tau}_{odp}(b, r^l)}\},$$

$$\forall p \in P(b, o, d, \tau), p \neq p^*; \tag{16a}$$

$$r^{\tau,l+1}_{odp^*}(b) = r^{\tau,l}_{odp^*}(b) + \sum_{p \in P(b,o,d,\tau), p \neq p^*} \rho^l \times \frac{r^{\tau,l}_{odp}(b) \times [GC^{\tau}_{odp}(b, r^l) - GC^{\tau}_{odp^*}(b, r^l)]}{GC^{\tau}_{odp}(b, r^l)}, \tag{16b}$$

where step size ρ^l is determined by the scheme of mixed step sizes, described in the following

$$\rho^l = 1/k, \text{ if } l = 0; \ \rho^l = 1, \text{ otherwise.} \tag{17}$$

This multi-class path assignment updating scheme is intuitively based on the fact that travelers farther from the equilibrium and on paths with larger flow rates are more inclined to change path than those on paths with smaller flow rates and with travel cost closer to the minimal cost.

5. NUMERICAL EXPERIMENTS

A set of numerical experiments is conducted to compare the solution quality of the proposed IMDUE algorithm (termed CG for column generation) and that of the MSA-based algorithm, developed by Lu et al. (2008), in addition to examining the convergence pattern and solution quality of the new algorithm. Both algorithms are coded and compiled by using the Compaq Visual FORTRAN 6.6 and evaluated on the Windows XP platform and a machine with an Intel Pentium IV 2.8 GHz CPU and 2GB RAM.

In all the experiments conducted, the following parameter settings are applied. The continuous VOT distribution considered in the experiments is a normal distribution with (mean, standard deviation) = (24, 12), denoted as N(24, 12). The parameters of this normal distribution are adapted from the estimated measurements in a value pricing experiment conducted in Southern California, USA (Brownstone and Small, 2005), and the unit of VOT in this study is United States dollars (USD) per hour. The feasible range of the VOT distribution $[\alpha^{\min}, \alpha^{\max}]$ is [0.6, 180].

A strict convergence criterion is used in the inner loop of the column generation-based algorithm; that is $|Gap(r^l) - Gap(r^{l-1})|/Gap(r^l) \leq 0.001$. The initial solutions are obtained by loading time-varying O-D demands to the extreme non-dominated paths calculated based on prevailing travel times output from the traffic simulator. Another measure of effectiveness (MOE), *AGap*(*r*), is also collected in the conducted experiments, in addition to *Gap*(*r*).

$$AGap(r) = \frac{\sum_b \sum_o \sum_d \sum_\tau \sum_{p \in P(b,o,d,\tau)} r^\tau_{odp}(b) \times [GC^\tau_{odp}(b,r) - GC^\tau_{odp*}(b,r)]}{\sum_b \sum_o \sum_d \sum_\tau \sum_{p \in P(b,o,d,\tau)} r^\tau_{odp}(b)} \tag{18}$$

This MOE, which is the average gap over all vehicles in the network for a given path flow pattern *r*, is independent of problem sizes and thus useful for examining the convergence pattern and solution quality of the proposed algorithm on different networks. The minimum of the *AGap*(*r*) is zero. Essentially, the smaller the average gap, the closer the solution is to the IMDUE.

5.1 Experiments on Irvine network

The Irvine (California, USA) network consists of 326 nodes (70 of them are signalized), 626 links, and 61 traffic analysis zones (TAZ). This network had been calibrated by using real-world observations from multiple-day detector data (Mahmassani et al., 2003). A 2-hour (7-9AM) morning peak time-varying O-D demand table is extracted from a 6-hour (4-10AM) demand table and loaded to the test network, with 35,300 vehicles in the observation period (7:10-8:50AM). To create hypothetical dynamic pricing scenarios, one lane of a portion (about 1 mile) of the I-405 westbound freeway is converted to a toll road, along with an additional new toll lane. The two toll lanes have the same length as the (remaining) three regular lanes but a 10-mile higher posted speed limit (and hence higher capacity) than the regular lanes. Table 1 lists the three simple dynamic pricing scenarios tested in the experiment conducted on the Irvine network. These three pricing scenarios have the same four pricing periods but different toll levels, each representing low, middle, and high toll scenarios, respectively.

Table 1 Dynamic road pricing scenarios tested on Irvine network

Pricing Scenario	Period 1 (7:00-7:30AM)	Period 2 (7:30-8:00AM)	Period 3 (8:00-8:30AM)	Period 4 (8:30-9:00AM)
1 (Low)	$0.10	$0.20	$0.30	$0.15
2 (Middle)	$0.20	$0.30	$0.40	$0.25
3 (High	$0.30	$0.40	$0.50	$0.35

The convergence patterns in terms of iteration-by-iteration gap values of the CG algorithm under the three dynamic pricing scenarios are presented in Table 2. It can be seen that the algorithm can effectively reduce the gap measure as well as the average gap in all three pricing scenarios tested on the Irvine network, although the convergence patterns are not strictly monotonic decreasing. As for the solution quality, the final gap values obtained by the new algorithm are 3.9% (196.3/5028.6), 4.5% (234.9/5211.2), and 5.4% (315.1/5795.7) of the initial gap values, respectively, for the three pricing scenarios. In addition, the average gap values for the three pricing scenarios, obtained by dividing these final gap values by the number of vehicles loaded in the observation period, are all less than 0.01 minutes. These small gap and average gap values indicate that the proposed algorithm is able to find close-to-IMDUE solutions for this network application.

Table 2 Convergence patterns of the CG algorithm on Irvine network

	Gap(r)			AGap(r)		
Iteration	Scenario 1	Scenario 2	Scenario 3	Scenario 1	Scenario 2	Scenario 3
0	5028.6	5211.2	5795.7	0.142	0.148	0.164
1	835.0	1025.6	851.3	0.024	0.029	0.024
2	787.1	892.2	822.7	0.022	0.025	0.023
3	452.8	624.6	546.9	0.013	0.018	0.015
4	536.9	505.0	501.4	0.015	0.014	0.014
5	590.7	597.7	407.3	0.017	0.017	0.012
6	376.1	415.4	542.2	0.011	0.012	0.015
7	409.6	332.2	419.5	0.012	0.009	0.012
8	523.4	342.0	385.8	0.015	0.010	0.011
9	316.2	369.4	366.9	0.009	0.010	0.010
10	406.5	357.9	299.1	0.012	0.010	0.008
11	372.9	280.1	460.6	0.011	0.008	0.013
12	430.7	294.8	402.2	0.012	0.008	0.011
13	335.7	238.9	237.7	0.010	0.007	0.007
14	589.1	256.4	292.6	0.017	0.007	0.008
15	274.5	255.4	320.2	0.008	0.007	0.009
16	283.4	252.9	353.9	0.008	0.007	0.010
17	271.2	228.3	249.3	0.008	0.006	0.007
18	247.1	268.3	323.7	0.007	0.008	0.009
19	258.4	285.3	313.0	0.007	0.008	0.009
20	196.3	234.9	315.1	0.006	0.007	0.009

The comparison of solution quality of CG and MSA is reported in Table 3 and Fig. 2. As shown in the figure and table, the proposed algorithm, CG, obtains a much better solution in terms of *Gap*(r) and *AGap*(r). The final *AGap*(r) value obtained by MSA (0.056 min) is 8 times larger than that obtained by CG (0.007 min). The gap reduction percentages of CG and MSA are 96% ((5211.2–234.9) / 5211.2) and 62% ((5211.2–1985.1) / 5211.2), respectively. The computation times of CG and MSA on Irvine network are about 24 and 23 hours, respectively. Thus, CG reduces the initial gap value 34% (=96%–62%) more than MSA with just 4.3% more of computation time.

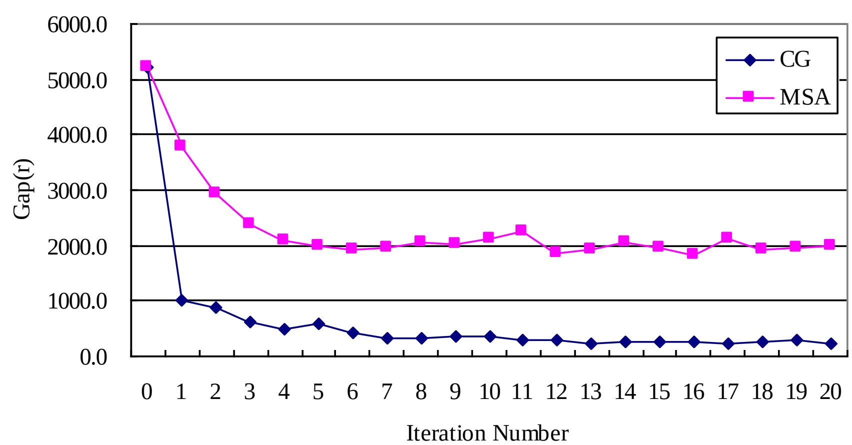

Fig. 2 Comparison of solution quality on Irvine network

Table 3 Comparison of solution quality on Irvine network

	CG		MSA	
Iteration	*Gap*(*r*)	*AGap*(*r*)	*Gap*(*r*)	*AGap*(*r*)
0	5211.2	0.148	5211.2	0.148
1	1025.6	0.029	3781.3	0.107
2	892.2	0.025	2925.5	0.083
3	624.6	0.018	2368.5	0.067
4	505.0	0.014	2102.6	0.060
5	597.7	0.017	1985.7	0.056
6	415.4	0.012	1934.1	0.055
7	332.2	0.009	1972.3	0.056
8	342.0	0.010	2053.4	0.058
9	369.4	0.010	2029.5	0.057
10	357.9	0.010	2130.6	0.060
11	280.1	0.008	2244.4	0.064
12	294.8	0.008	1851.9	0.052
13	238.9	0.007	1923.5	0.054
14	256.4	0.007	2045.4	0.058
15	255.4	0.007	1959.2	0.056
16	252.9	0.007	1831.8	0.052
17	228.3	0.006	2117.5	0.060
18	268.3	0.008	1921.0	0.054
19	285.3	0.008	1952.3	0.055
20	234.9	0.007	1985.1	0.056

5.2 Experiments on CHART network

To further demonstrate the capability of the proposed algorithm for large-scale networks with dynamic road pricing scenarios, another experiment is conducted on a large road network: the Maryland CHART network, which consists primarily of the I-95 freeway corridor between Washington, D.C. and Baltimore (Maryland, USA) and is bounded by two beltways (I-695 Baltimore Beltway on the north and I-495 Capital Beltway on the south). The CHART network has 2241 nodes (231 of them are signalized), 3459 links and 111 traffic analysis zones (TAZ). This network had been calibrated by using real-world observations from multiple-day detector data (Mahmassani et al. 2006). An available 1-hour (7:30-8:30AM) morning peak time-varying O-D demand (with 39,560 vehicles in the observation period from 7:40 to 8:20 AM) table is extracted and loaded to the network. To create hypothetic dynamic toll scenarios, one of the 20-mile long southbound lanes of the I-95 corridor is converted to the toll road, together with an additional new toll lane. The two toll lanes have the same length, posted speed limit, and capacity as the (remaining) three regular lanes. The two-lane toll road consists of 57 links in the coded network, and the four access/egress points to/from the toll road are interchanges with I-195, MD-100, MD-32 and MD-198, where additional on-ramps and off-ramps are added. A dynamic link toll vector generated by the method proposed by Dong et al. (2011) is used in this network to test the BDUE algorithm.

The comparison of solution quality of CG and MSA is reported in Table 4 and Fig. 3. As presented in the figure and table, CG significantly outperforms MSA, because the final *AGap*(*r*) value obtained by MSA (0.149 min) is almost 19 times larger than that obtained by CG (0.008 min). The gap reduction percentages of CG and MSA are 98.8% ((25393.2−300.1) / 25393.2) and

76.7% ((25393.2−5915.4) / 25393.2), respectively. The computation times of CG and MSA on CHART network are about 40 and 39 hours, respectively. Similar to the result found on Irvine network, CG reduces the initial gap value 22% (=98.8%−76.7%) more than MSA with just 2.5% more of computation time. While MSA was reported in the previous work (e.g., Lu et al., 2008) to be able to find acceptable solutions, the results on both networks shown in the current paper demonstrate that CG is more effective than MSA in obtaining close-to-IMDUE solutions. Note that for planning purpose, those amounts of computation time are acceptable, and less computation time can be achieved with more powerful machines and/or more efficient implementation (coding) of the algorithm.

Table 4 Comparison of solution quality on CHART network

	CG		MSA	
Iteration	*Gap*(*r*)	*AGap*(*r*)	*Gap*(*r*)	*AGap*(*r*)
0	25393.2	0.641	25393.2	0.641
1	5622.9	0.142	11082.9	0.280
2	3207.4	0.081	13210.4	0.334
3	1885.5	0.048	11933.5	0.301
4	1379.7	0.035	9847.8	0.249
5	1051.3	0.027	8325.4	0.210
6	984.0	0.025	7775.6	0.196
7	814.4	0.021	6397.9	0.162
8	816.0	0.021	6072.8	0.153
9	621.6	0.016	6591.1	0.166
10	642.8	0.016	7178.3	0.181
11	492.1	0.012	6325.7	0.160
12	427.8	0.011	5637.8	0.142
13	415.8	0.011	6104.7	0.154
14	310.2	0.008	6389.6	0.161
15	300.1	0.008	5915.4	0.149

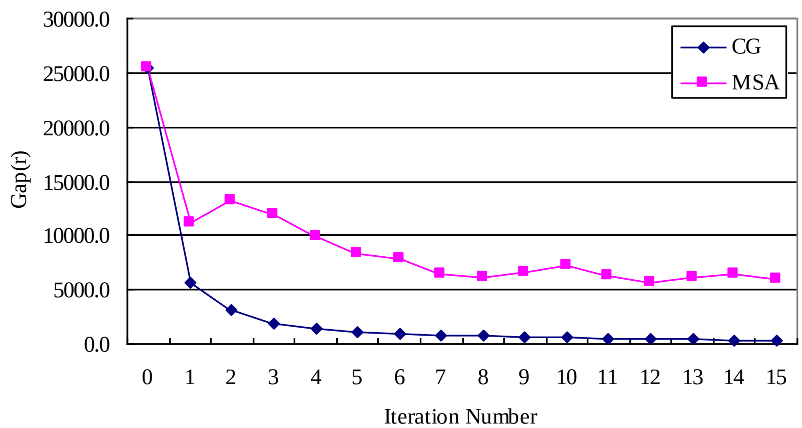

Fig. 3 Comparison of solution quality on CHART network

6 Concluding Remarks

This paper presents an efficient algorithm for solving the IMDUE problem, or specifically the BDUE problem, which assumes the VOT is continuously distributed across the population of trips. The proposed column generation-based algorithm (i) applies the parametric analysis method in the outer loop to determine multiple user classes and to generate representative extreme non-dominated paths, and (ii) solves FMDUE problems in the inner loop by a feasible descent direction method. Although the mathematical abstraction of the problem is a typical analytical formulation, the solution algorithm adopts the simulation-based approach to tackle many practical aspects of the DTA applications. The experimental results show that the convergence pattern of the proposed algorithm is not affected by the different VOT assumptions (constant or random VOT), and the algorithm is able to find close-to-IMDUE (or approximate) solutions. In addition, the solution quality of the proposed algorithm is much better than that of the MSA-based solution method.

Several interesting research directions can be continued based on the rich modeling capabilities of the IMDUE model in capturing traffic dynamics and user heterogeneity. For instance, the model can be extended to consider O-D-specific and/or time-varying VOT distributions, provided that the data are available to estimate the underlying parameter distributions. The model can also be integrated into a solution framework aiming at finding optimal or Pareto-improving dynamic pricing schemes, including locations, pricing periods and toll charges, so as to alleviate congestion. In addition, incorporating stochastic path choice with explicit perception errors (e.g. logit or probit models) would be an important and interesting extension. It will be interesting to implement some other VI algorithms developed for the static traffic assignment problems and compare their performance with that of our proposed approach. The model presented in this paper can be viewed as laying the foundation for a platform that integrates more realistic behavioral modeling in a dynamic network analysis tool. The main challenges in this development is to continue pushing the boundary of what can be realistically handled in a large network setting within the limits of practical computational capabilities.

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APPENDIX

Proof of Proposition 1:

Suppose r^* satisfies the MDUE condition, and let $G(r^*)$ be the corresponding path generalized cost vector. According to the MDUE definition, for an equilibrium multi-class path flow vector r^*, the following condition can be established:

If $G_{odq}^{\tau}(\alpha, r^*) > G_{odp}^{\tau}(\alpha, r^*) \,(= \pi_{od}^{\tau}(\alpha))$, then $r_{odq}^{\tau}(\alpha) = 0$.

If $G_{odq}^{\tau}(\alpha, r^*) = G_{odp}^{\tau}(\alpha, r^*) \,(= \pi_{od}^{\tau}(\alpha))$, then $r_{odq}^{\tau}(\alpha) \geq 0$.

$\forall o, d, \tau, p$ and $q \in P(o, d, \tau)$, and $\forall \alpha \in [\alpha^{\min}, \alpha^{\max}]$ (A1)

where $\pi_{od}^{\tau}(\alpha)$ is the minimum possible generalized travel cost for the trips with VOT α from o to d departing at time τ.

Consider these path generalized costs $G(r^*)$ as fixed at the current level of path flow r^*. Because r^* satisfies the MDUE condition and hence only least generalized cost paths are used, the total generalized cost cannot be reduced by moving flows from least generalized cost paths to other inefficient paths. For instance, if path generalized costs are fixed as $G(r^*)$ and $G_{odq}^{\tau}(\alpha, r^*)$ > $G_{odp}^{\tau}(\alpha, r^*)$ then moving flow $r_{odp}^{\tau}(\alpha)$ from p to q will lead to an increase of total generalized cost by $r_{odp}^{\tau}(\alpha) \times G_{odq}^{\tau}(\alpha, r^*) - r_{odp}^{\tau}(\alpha) \times G_{odp}^{\tau}(\alpha, r^*) > 0$. Therefore any other feasible multi-class path flow vector $r \in \Omega$ has total generalized cost at least as large as r^* which uses only cheapest paths. In other words, the MDUE path flow vector r^* satisfying (A1) is also the solution of the infinite dimensional variational inequality (8):

$G(r^*)^T \circ r^* \leq G(r^*)^T \circ r$ or $G(r^*)^T \circ (r^* - r) \leq 0, \forall\, r \in \Omega$ (A2)

Conversely, assume that condition (A1) does not hold, then there exists the following situation for some triplet (o, d, τ) and paths p and q: $r_{odq}^{\tau}(\alpha) > 0$ and $G_{odq}^{\tau}(\alpha, r^*) > G_{odp}^{\tau}(\alpha, r^*)$ $(= \pi_{od}^{\tau}(\alpha))$. Moving flow $r_{odq}^{\tau}(\alpha)$ from q to the cheaper path p will result in a reduction of total generalized cost by $r_{odq}^{\tau}(\alpha) \times G_{odq}^{\tau}(\alpha, r^*) - r_{odq}^{\tau}(\alpha) \times G_{odp}^{\tau}(\alpha, r^*) > 0$. Let the resulting flow pattern be $r \in \Omega$. Then $G(r^*)^T \circ r < G(r^*)^T \circ r^*$, in which case Eq.(8) is not satisfied.

From above, if (A1) is satisfied then Eq.(8) is satisfied, and if (A1) does not hold then Eq.(8) does not, either. Thus, conditions (A1) and Eq.(8) are equivalent, and solving for the IMDUE flow pattern is equivalent to finding the solution of the (possibly) infinite dimensional variational inequality Eq.(8). This completes the proof.

Proof of Proposition 2:

Denote by ϕ_a^t the time-dependent link marginal travel time: the travel time contribution of an additional unit of vehicular flow on link a in time interval t to the link travel time d_a^t. By assuming that d_a^t is a monotonic (increasing) function of x_a^t (the number of vehicles on link a in time interval t): $[(x_a^t + \Delta x_a^t) - x_a^t] \times [d_a^t(x_a^t + \Delta x_a^t) - d_a^t(x_a^t)] \geq 0$ (e.g. Nagurney, 1998), with $\Delta x_a^t > 0$, the following can be obtained:

$$\phi_a^t = \lim_{\Delta x_a^t \to 0} \frac{d_a^t(x_a^t + \Delta x_a^t) - d_a^t(x_a^t)}{\Delta x_a^t} \geq 0. \tag{A3}$$

Note that this study considers ϕ_a^t as a local link marginal. Peeta (1994) gave a comprehensive discussion on global link marginals with temporal and spatial interactions.

Since link costs (i.e. tolls) c_a^t are given as input, the path marginal generalized cost perceived by the tripmaker of user class *u*(*b*), $\phi_{odp}^{\tau}(b,r)$, is assumed to be the sum of constituent link marginal travel times weighted by the VOT of user class *u*(*b*).

$$\eta_{odp}^{\tau}(b,r) = \sum_{a \in A(p)} \phi_a^t \times \delta_{odpa}^{\tau,t} \times \alpha(b) \tag{A4}$$

where *A*(*p*) is the set of links on path *p*, *t* is the first time interval in which link *a* on path *p* is reached by a vehicle assigned to that path in time τ, and $\delta_{odpa}^{\tau,t}$ is the time-dependent link-path incidence indicator; $\delta_{odpa}^{\tau,t}$ =1 if vehicles going from *o* to *d* assigned to path *p* at time τ pass link *a* in time interval *t*, and 0 otherwise.

Recall that *p** be the referenced shortest path for a (*b*, *o*, *d*, τ). Then constraints (14*c*) can be re-written as the following:

$$r_{odp^*}^{\tau}(b) = h_{od}^{\tau}(b) - \sum_{p \in P(b,o,d,\tau) \setminus p^*} r_{odp}^{\tau}(b), \ \forall b,\ o,\ d, \text{ and } \tau. \tag{A5}$$

Define a new path flow vector $y = \{y_{odp}^{\tau}(b), \forall p \in P(b,o,d,\tau) \setminus p^*, \forall b,o,d,\tau\}$. By substituting Eq.(A3) into the objective function (14*a*), the NMP becomes the following unconstrained minimization problem:

$$Min\, Gap(y) = \sum_b \sum_o \sum_d \sum_\tau \sum_{p \in P(b,o,d,\tau) \setminus p^*} y_{odp}^{\tau}(b) \times [GC_{odp}^{\tau}(b,y) - GC_{odp^*}^{\tau}(b,y)] \tag{A6}$$

Note that the constraints (14*b*) and (14*d*) are satisfied in the NMP because of the aforementioned active constraint set strategy and the projection of the updated solution onto the feasible set Ω, respectively. With this transformation and according to Eq.(A4), the first-order partial derivative of *Gap*(*r*) with respect to a particular $y_{odp}^{\tau}(b)$ is obtained as the following:

$$\begin{aligned} \frac{\partial Gap(y)}{\partial y_{odp}^{\tau}(b)} &= GC_{odp}^{\tau}(b,y) - GC_{odp^*}^{\tau}(b,y) + y_{odp}^{\tau}(b) \times \frac{\partial [GC_{odp}^{\tau}(b,y) - GC_{odp^*}^{\tau}(b,y)]}{\partial y_{odp}^{\tau}(b)} \\ &+ \sum_{p' \in P(b,o,d,\tau) \setminus (p^* \cup p)} y_{odp'}^{\tau}(b) \times \frac{\partial [GC_{odp}^{\tau}(b,y) - GC_{odp^*}^{\tau}(b,y)]}{\partial y_{odp}^{\tau}(b)} \\ &= GC_{odp}^{\tau}(b,y) - GC_{odp^*}^{\tau}(b,y) + y_{odp}^{\tau}(b) \times \sum_{a \in B(p)} \phi_a^t \times \alpha(b) \\ &+ \sum_{p' \in P(b,o,d,\tau) \setminus (p^* \cup p)} (y_{odp'}^{\tau}(b) \times \sum_{a \in A(p') \cap B(p)} \phi_a^t \times \alpha(b)) \end{aligned} \tag{A7}$$

where $A(p)$ is the set of links on path p, and $B(p)=\overline{A(p)\cap A(p^*)}$ is the set of links that are on either the non-shortest path p or the referenced shortest path p^*. In the following, we prove that the search direction $[y\frac{GC(y)-GC^*(y)}{GC(y)}]$ is a descent direction of $Gap(y)$ at y.

To prove the vector Dir = $[y\frac{GC(y)-GC^*(y)}{GC(y)}]$ is a descent direction of $Gap(y)$ at y, it is necessary to show that the inner product $\nabla Gap(y) \bullet (-1 \times Dir) < 0$ (see e.g. Theorem 4.1.2 in Bazara'a et al. 1993). Component-wise, this is equivalent to showing that

$$\sum\nolimits_{p\in P(b,o,d,\tau)\backslash p^*}((-1)\times\frac{\partial Gap(y)}{\partial y^{\tau}_{odp}(b)}\times Dir^{\tau}_{odp}(b))<0,\ \forall b,\ o,\ d, \text{ and } \tau, \tag{A8}$$

where $Dir^{\tau}_{odp}(b)=[y^{\tau}_{odp}(b)\times\frac{(GC^{\tau}_{odp}(b,y)-GC^{\tau}_{odp^*}(b,y))}{GC^{\tau}_{odp}(b,y)}]$ and $\frac{\partial Gap(y)}{\partial y^{\tau}_{odp}(b)}$ is defined as Eq.(A7).

Consider that, for a (b, o, d, τ) and for each path p ($r^{\tau}_{odp}(b)$>0) in the path set $P(b, o, d, \tau)\backslash p^*$, the cost of path p could be either equal to or greater than the least cost . In the first case, p is one of the shortest (more precisely, least cost) paths, then $Dir^{\tau}_{odp}(b)=0$ and accordingly $(-1)\times\frac{\partial Gap(y)}{\partial y^{\tau}_{odp}(b)}\times Dir^{\tau}_{odp}(b)$ =0. In the latter case, p is a non-shortest path, then $Dir^{\tau}_{odp}(b)>0$. According to Eq.(A3), link marginal travel times are non-negative and $GC^{\tau}_{odp}(b,y)-GC^{\tau}_{odp^*}(b,y)$ is positive for any non-shortest path p, so $\frac{\partial Gap(y)}{\partial y^{\tau}_{odp}(b)}>0$ and $(-1)\times\frac{\partial Gap(y)}{\partial y^{\tau}_{odp}(b)}\times Dir^{\tau}_{odp}(b)<0$.

Mathematically, for each (b, o, d, τ)

$$\begin{aligned}&\sum\nolimits_{p\in P(b,o,d,\tau)\backslash p^*}[(-1)\times\frac{\partial Gap(y)}{\partial y^{\tau}_{odp}(b)}\times Dir^{\tau}_{odp}(b)]\\&=-\sum\nolimits_{p\in P_{\pi}(b,o,d,\tau)\backslash p^*}[\frac{\partial Gap(y)}{\partial y^{\tau}_{odp}(b)}\times Dir^{\tau}_{odp}(b)]-\sum\nolimits_{p\in P(b,o,d,\tau)\backslash P_{\pi}(b,o,d,\tau)}[\frac{\partial Gap(y)}{\partial y^{\tau}_{odp}(b)}\times Dir^{\tau}_{odp}(b)]\\&=0-\sum\nolimits_{p\in P(b,o,d,\tau)\backslash P_{\pi}(b,o,d,\tau)}[\frac{\partial Gap(y)}{\partial y^{\tau}_{odp}(b)}\times Dir^{\tau}_{odp}(b)]<0\end{aligned} \tag{A9}$$

where $P_{\pi}(b, o, d, \tau)$ is the set of least generalized cost paths of a (b, o, d, τ). Thus the search direction Dir is a descent direction of $Gap(y)$ at y. This completes the proof.