DEVELOPMENT OF OPTIMAL NETWORK DESIGN METHOD IN BI-LEVEL PROGRAMMING USING SENSITIVITY OF VARIABLE TRAVEL DEMAND

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Abstract: This paper is concerned with a bi-level model formulation for the equilibrium network design problem with variable travel demand. The lower level problem is a variable demand network equilibrium model that describes user's route choice behavior for a given set of link capacity enhancements. In the case with variable demands in the lower level problem, to choose the minimization of total travel cost as the objective function of the upper problem may result in an impractical solution because this might be achieved through minimizing travel demand. In this paper the upper level problem determines an optimal set of link capacity enhancements to optimize a system performance measure, or the increase of net user net benefit measured by consumer's surplus, while taking account of the limited budget constraint. For the solution of the suggested bi-level program sensitivity analysis based(SAB) algorithm is used to obtain the best set of network links that should be considered first for improvement.

Key Words: Equilibrium Network Design, Bi-level model, Variable Travel Demand, Net User Benefit, Sensitivity Analysis

1. INTRODUCTION

The equilibrium network design problem(ENDP) involves the selection of new facilities(links) to add to a transportation network or the determination of a set of capacity enhancements for some existing links so that the system performance and capital investment costs are optimal when the network flow pattern is constrained to be a user equilibrium.
Conventional studies on ENDP with fixed travel demand models assume that the estimated future OD travel demand might not be changed even if the structure and the capacity of the network are improved. However the OD travel demand actually shifts with the network service level and the fixed demand assumption may lose its validity in the long-range strategic network design. Thus, it is desirable to involve the variable travel demand which is

determined endogenously in the model in the optimal network design(Friesz T.L., 1985). Y. Asakura and T. Sasaki(1990) formulated the optimal network design problem incorporating the variable travel demand by using the bilevel programming framework and tested the feasibilities of the formulated ENDP for the application to the actual road network planning. The user Equilibrium problem with variable demand is employed as the lower problem of the NDP for network flow description and total travel cost minimization the upper problem.

This paper presents a bilevel model formulation for ENDP with variable travel demand. The lower level problem is a variable demand network equilibrium model that describes user's route choice behavior for a given set of link capacity enhancements. The upper level problem determines an optimal set of link capacity enhancements to optimize a system performance measure, such as net user benefit measured by consumer's surplus, while taking account of the limited budget constraint.

However it might be difficult to solve for a global optimum in the bilevel mathematical programming problem because of its non-convexity(Friesz, T. L. et al., 1990). Several algorithms have been proposed for solving non-convex bilevel network design problem, but here sensitivity analysis based(SAB) algorithm is adopted in the solution algorithm. It has been recently applied to optimal ramp metering in general freeway-arterial corridor systems(H. Yang., S. Yagar, 1994), traffic signal control in saturated road networks(H. Yang., S. Yagar, 1995). And H. Yang(1997) proposed the applicability of sensitivity analysis for the variable demand network equilibrium problem to the road network design. The results of the sensitivity analysis are the derivatives of the equilibrium link flow pattern and OD travel demand when the network characteristics such as link capacities are changed slightly. The derivative information from the sensitivity analysis could help finding the most sensitive link to the network equilibrium flow pattern under slight link capacity enhancement, which should be considered first to improve in the current road network.

2. BILEVEL MODEL FORMULATION AND SOLUTION ALGORITHM

2.1 Net User Benefit

It should be noted that in the case with variable demands, to choose the minimization of total travel cost as the objective function of the upper problem may result in an impractical solution because this might be achieved through minimizing travel demand. That is, lower and less investment can minimize the travel demand and the total travel cost. On the other hand, since the travel cost of induced demand can outweigh the cost reduction that is brought by improvements in network capacity, the total travel cost might not be decreasing monotonously.

Since the induced OD trips after the link capacity increase would bring benefits the network users, one appropriate objective function is to maximize the net user benefit derived from consumer's surplus which is the user benefit minus user cost. So in this paper in order to evaluate the user benefit as well as the user cost from the increased OD travel demand we now consider how to measure the net user benefits to be derived from the improvement in link capacities.

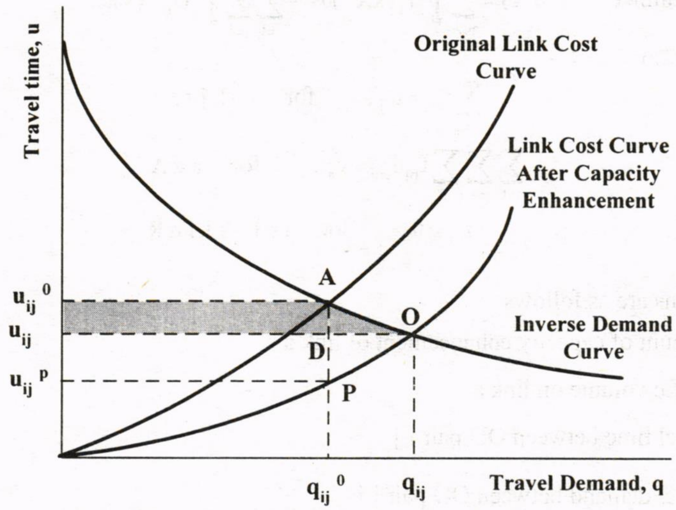

Fig. 1 Relationship between travel demand and cost

In Fig. 1 before the capacity enhancement the equilibrium flow and user cost is determined by the intersection of the original link cost curve and demand curve at A. If traffic flow conditions are shifted from the equilibrium position A to O by the capacity enhancement, then the change in user benefit will be represented by the shaded areas which is the sum of area $Au_{ij}^{o}u_{ij}D$ and area ΔADO. The former represents the cost reduction of the previous network users due to improved speed and the latter is the consumer's surplus enjoyed by induced network users. Thus the increase in the net user benefit from the enhancement of link capacity is calculated as follows

$$NB = \sum_{i \in I}\sum_{j \in J} u_{ij}^{o} q_{ij}^{o} - \sum_{i \in I}\sum_{j \in J} u_{ij}\, q_{ij} - \sum_{i \in I}\sum_{j \in J} \int_{q_{ij}}^{q_{ij}^{o}} D_{ij}^{-1}(x)dx \tag{2.1}$$

2.2 Formulation of Equilibrium Network Design Problem

The equilibrium network design problem considered here is to maximize the increase of net user benefit which results from a set of link capacity enhancements within the budget constraints, while the OD travel demands and times are obtained by solving the lower level network equilibrium problem with variable demand.

Maximize $\quad NB(q(c), u(c))$

$$= \sum_{i \in I}\sum_{j \in J} u_{ij}^{o} q_{ij}^{o} - \sum_{i \in I}\sum_{j \in J} u_{ij}\, q_{ij} - \sum_{i \in I}\sum_{j \in J} \int_{q_{ij}}^{q_{ij}^{o}} D_{ij}^{-1}(x)dx \tag{2.2}$$

sub. to

$$\sum_{a \in A^*} G_a(c_a) \le G \tag{2.3}$$

$$C_a \ge 0 \qquad \text{for} \quad a \in A^* \tag{2.4}$$

where $q(c)$ and $u(c)$ are determined by solving:

$$\text{Minimize} \quad F(q,v)=\sum_{a\in A}\int_0^{v_a} t_a(x,c_a)dx-\sum_{i\in I}\sum_{j\in J}\int_0^{q_{ij}} D_{ij}^{-1}(x)dx \tag{2.5}$$

sub. to

$$\sum_{r\in R_{ij}} f_{rij}=q_{ij}, \qquad \text{for} \quad i\in I, j\in J \tag{2.6}$$

$$\sum_{i\in I}\sum_{j\in J}\sum_{r\in R_{ij}} f_{rij}\delta_{arij}=v_a, \qquad \text{for} \quad a\in A \tag{2.7}$$

$$f_{rij}\geq 0, \qquad \text{for} \quad i\in I, j\in J, r\in R_{ij} \tag{2.8}$$

Used notations are as follows

c_a : amount of capacity enhancement of link a

v_a : traffic volume on link a

u_{ij} : travel time between OD pair i-j

q_{ij} : travel demand between OD pair i-j

R_{ij} : set of paths between OD pair i-j

f_{rij} : traffic flow on path r connecting OD pair i-j

δ_{arij} : 1 if path r uses link a, and 0 otherwise

$D_{ij}^{-1}(q_{ij})$: inverse demand function between OD pair i-j

$G_a(c_a)$: investment cost function of link a

G : upper limit of investment cost

$t_a(v_a, c_a)$: link cost function $\left(= t_{a0}\left\{1+r\left(v_a/(K_a+c_a)\right)^k\right\}\right)$ (BPR function)

t_{a0} : free-flow travel time of link a

K_a : capacity of link a

A : set of links in the network

A^* : subset of capacity enhancement links

c, v, u, q : set of each variable

The decision variable of above model is the individual link capacity and it is assumed to be continuous. Various methods can be adopted to improve link capacity based on the network characteristics, such as widening lane widths or lateral clearance, signal coordination, parking controls and so on.

2.3 Sensitivity Analysis for Network Equilibrium Problem

If we assume that perturbation parameters, exist in the link cost function and OD travel demands and $t(v,\varepsilon)$ and $D(u,\varepsilon)$ are once continuously differentiable in, then the general perturbed network equilibrium problem can be formulated as the following perturbed variational inequality.

Since the variational inequality formulation of the problem usually involves path flow variables, direct application of variational inequality sensitivity analysis to the perturbed equilibrium traffic assignment problem is not feasible because the path flow pattern is not usually unique at equilibrium assignment problems. In order to overcome the nonuniqueness difficulty, Tobin and Friesz(1988) proposed a restriction approach for sensitivity analysis of network equilibrium problems.

Find $(v^*, q^*) \in \Omega$ such that

$$t(v^*,\varepsilon)\cdot(v-v^*)-D^{-1}(q^*,\varepsilon)^T\cdot(q-q^*)\geq 0 \qquad (2.9)$$

for all $v, q \in$

where $\Omega = \{(v,q)|v=\Delta f, q=\Lambda f, f\geq 0)$

where Δ is link/path incidence matrix

Λ is OD/path incidence matrix

The following results are well summarized by H. Yang(1997) and are presented here without proof.

The restriction approach is to select a nondegenerate extreme point in the feasible region of equilibrium path flows. An extreme point can be obtained easily if the Frank-Wolfe algorithm is used to solve the network equilibrium problem. The Frank-Wolfe algorithm generates a unique set of minimum time paths between each OD pair at each iteration. If the paths generated are saved from iteration to iteration, upon termination the algorithm provides an equilibrium path flow pattern and a link/path incidence matrix for the paths used. A nondegenerate extreme point can then be selected from this set of equilibrium path flows which is defined as:

$$\Omega^* = \{(v^*,q^*)|v^*=\Delta f, q^*=\Lambda f, f\geq 0)\} \qquad (2.10)$$

When we consider only the nondegenerate extreme point of positive path flow solutions, f^*, the system of equation then reduces to:

$$\hat{t}^0(f^*,0)-\Lambda^{0T}u=0 \qquad (2.11)$$

$$\Lambda^0 f^0 * -D(u,0)=0 \qquad (2.12)$$

where ^{0} represents the corresponding reduced vectors and matrices.

Differentiating both sides of eqns (2.11) and (2.12) with respect to the perturbation parameter ε, we obtain

$$\begin{bmatrix}\nabla_\varepsilon f^0\\ \nabla_\varepsilon u\end{bmatrix}=\begin{bmatrix}\nabla_f \hat{t}^0(f^*,0) & -\Lambda^{0T}\\ \Lambda^0 & -\nabla_u D(u,0)\end{bmatrix}\cdot\begin{bmatrix}\nabla_\varepsilon \hat{t}^0(f^*,0)\\ \nabla_\varepsilon D(u,0)\end{bmatrix} \qquad (2.13)$$

Let

$$J_{f^0,u}(0)=\begin{bmatrix}\nabla_f\hat{t}^0(f^*,0) & -\Lambda^{0T}\\ \Lambda^0 & -\nabla_u D(u,0)\end{bmatrix}, \quad \left[J_{f^0,u}\right]^{-1}=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix} \tag{2.14}$$

The following can be easily obtained

$$B_{22}=\left[-\nabla uD(u,0)+\Lambda^0\nabla_f\hat{t}^0(f^*,0)^{-1}\Lambda^{0T}\right]^{-1} \tag{2.15}$$

$$B_{12}=\nabla_f\hat{t}^0(f^*,0)^{-1}\Lambda^{0T}B_{22} \tag{2.16}$$

$$B_{21}=-B_{22}\Lambda^0\nabla_f\hat{t}^0(f^*,0)^{-1} \tag{2.17}$$

$$B_{11}=\nabla_f\hat{t}^0(f^*,0)^{-1}\left[E+\Lambda^{0T}B_{21}\right] \tag{2.18}$$

where **E** is an identity matrix of appropriate dimension.

Table 3.1 shows the general expressions of the derivatives of decision variables(path flow, link flow, OD travel time, OD travel demand and net user benefit) with respect to a variety of perturbation parameters in demand functions and link cost functions.

Table 1 Derivatives of decision variables with respect to ε

Variable	Derivative
Path flow $(\nabla_\varepsilon f^0)$	$-B_{11}\nabla_\varepsilon\hat{t}^0(f^*,0)+B_{12}\nabla_\varepsilon D_u(u,0)$
Link flow $(\nabla_\varepsilon v)$	$-\Delta^0B_{11}\Delta^{0T}\nabla_\varepsilon\hat{t}^0(v^*,0)+\Delta^0B_{12}\nabla_\varepsilon D_u(u,0)$
OD travel time $(\nabla_\varepsilon u)$	$-B_{21}\Delta^{0T}\nabla_\varepsilon\hat{t}^0(v^*,0)+B_{22}\nabla_\varepsilon D_u(u,0)$
OD travel demand $(\nabla_\varepsilon q)$	$\nabla_\varepsilon D_u(u,0)+\nabla_u B_{12}D_u(u,0)\nabla_\varepsilon u$
Net User Benefit $(\nabla_\varepsilon NB)$	$-[\nabla_\varepsilon u]^T\cdot q$

2.4 Solution Algorithm

The sensitivity analysis based(SAB) algorithm is shown as follows

Step 0: *Initialization.* Determine an initial set of capacity enhancement $c^{(0)}$. Set $n=0$

Step 1: *Update* $q(c)$ *and* $v(c)$. Solve the lower level variable demand network equilibrium problem for given $c^{(n)}$ using Frank-Wolfe algorithm.

Step 2: *Perform sensitivity analysis.* Obtain the derivatives $\frac{\partial q}{\partial c}, \frac{\partial v}{\partial c}, \frac{\partial u}{\partial c}$

Step 3: *Solve the upper level problem* . The upper level objective function is linearly approximated to find the auxiliary solution, **u**, based on the derivative information in step 2. Given the initial solutions, $v(c_a^*)$, $q(c_a^*)$ at c_a^*, the objective function of upper level can be linearly approximated as follows after some manipulations.

$$\overline{NB}(c_a)=-\sum_{a\in A^*}\left(\sum_{i\in I}\sum_{j\in J}\frac{\partial u_{ij}}{\partial c_a}q_{ij}\right)\cdot c_a$$

The solution indicates the farthest point in the steepest direction of the upper level objective function within the budget constraint.

Step 4: *Determine step size.* Find α_n that solves

$$\alpha_n = \beta/(1+n)^{\gamma}, \quad (\beta, \gamma \text{ are parameters})$$

Step 5: *Update capacity enhancement.* $c^{(n+1)} = c^{(n)} + \alpha_n (u - c^{(n)})$

Step 6: *Convergence criterion.* $\left| c_a^{(n+1)} - c_a^{(n)} \right| \le \kappa$, stop. Otherwise set $n = n+1$ and go to step 1.

In the step size determination step, predetermined sequences of step length method is used which must satisfy the conditions, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$

3. APPLICATION TO HYPOTHETICAL NETWORK

3.1 Network Data

To illustrate the discussion in the previous sections let us consider the hypothetical road network(4 OD pairs, 13 nodes and 28 links including 4 dummy links) depicted in Fig. 2. For the analysis's convenience all the links are assumed to be one-directed. The directed link number is given above links in Fig. 2. The links are classified as three classes, and link characteristics of each class are same and presented in Table 3. The parameter values of link cost function(BPR function) are r=0.15, k=4. Initial candidates of capacity enhancement are represented as bold arrows in the network(Link No. 8,9,11,13,20,22,24,25).

The OD travel demand functions are assumed to be the following negative exponential functions.

$$D_{ij}(u_{ij}) = A_i B_j \exp(-\theta u_{ij})$$

where A_i and B_j are activity level at origin i and destination j, and θ is a parameter that should be estimated.

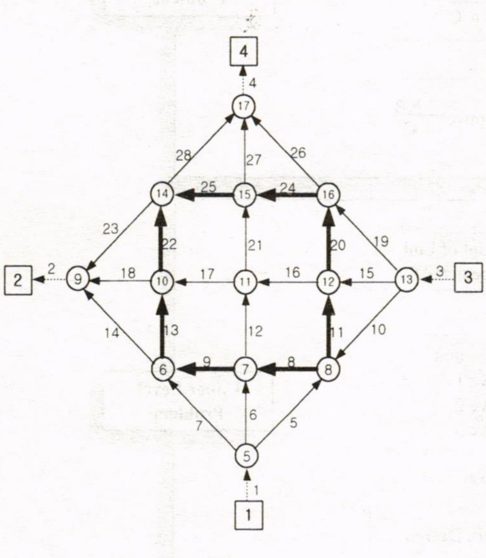

Fig. 2 Test Network

Table 2 OD demand function parameters

Origin	Destination	$A_i \cdot B_j$	θ
1	2	25000	0.15
1	4	45000	0.2
3	2	35000	0.2
3	4	20000	0.15

Table 3 Link characteristics of each road class

Road Class	Link No.	Lanes	Length (km)	Design Speed (kph)	Capacity (vph)
1	6,12,21,27,15,16,17,18	2	7	80	2200
2	8,9,11,13,20,22,24,25	1	8	70	2000
3	5,7,10,14,19,23,26,28	1	10	60	1800

The investment cost function in the upper level problem is assumed to be linear and three upper budget limits are considered.

With these network data the final output by Frank-Wolfe algorithm in the lower level problem includes a complete set of link flow patterns, a subset of minimum time paths used by drivers and the distribution of demand among these paths. This information is necessary for implementing sensitivity analysis.

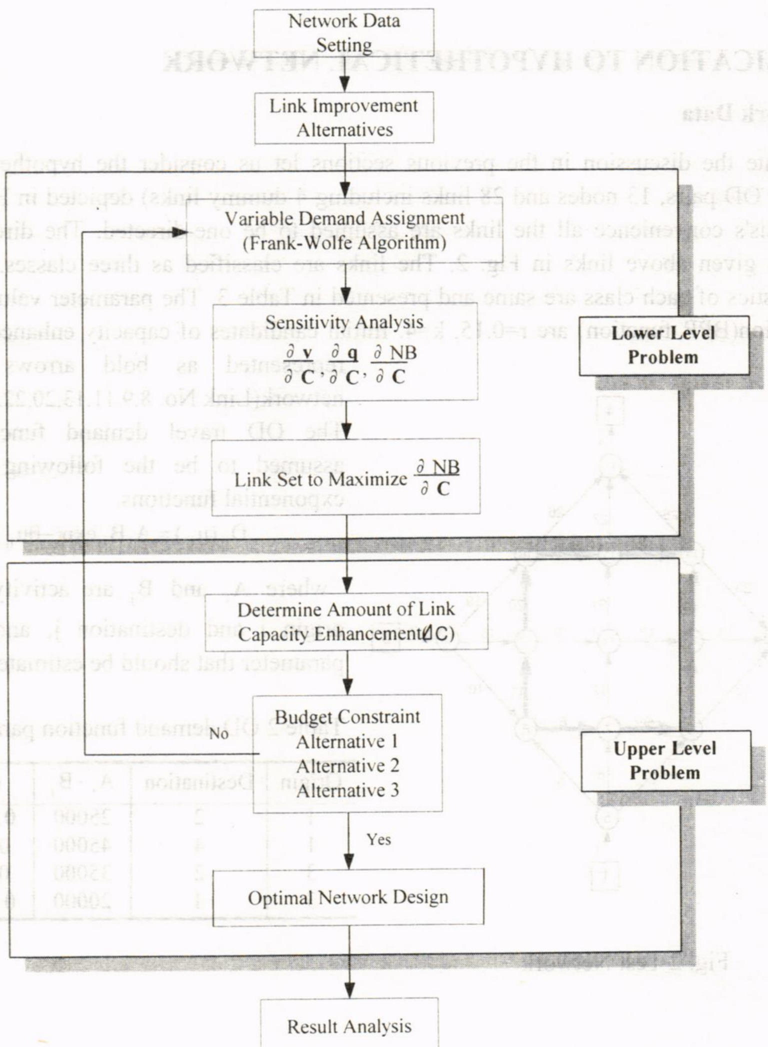

Fig. 3 Process of Analysis

3.2 Results of Sensitivity Analysis

From the results of sensitivity analysis at initial link capacities we can select the best alternative, i.e. which links among the initial 8 candidate links should be considered to improve first. In Table 4 and Table 5 derivatives of OD travel times, OD travel demands and upper level objective function(increase in net user benefit) with respect to each candidate link capacity are presented. The derivatives indicates the change in travel times, travel demands and net user benefit when there is a unit increase in the one of candidate link capacities.

Table 4 Derivatives of OD travel time at initial link capacities

OD travel time	$\partial(\bullet)/\partial K_8$	$\partial(\bullet)/\partial K_9$	$\partial(\bullet)/\partial K_{11}$	$\partial(\bullet)/\partial K_{13}$
u_{12}	5.21× 10E-5	-1.09× 10E-4	2.18× 10E-5	-9.12× 10E-5
u_{14}	1.97× 10E-5	-3.73× 10E-6	-1.01× 10E-4	-8.98× 10E-5
u_{34}	-8.85× 10E-5	-7.74× 10E-5	-5.18× 10E-5	-3.75× 10E-5
u_{32}	-5.09× 10E-5	-2.66× 10E-5	9.26× 10E-5	1.56× 10E-5
OD travel time	$\partial(\bullet)/\partial K_{20}$	$\partial(\bullet)/\partial K_{22}$	$\partial(\bullet)/\partial K_{24}$	$\partial(\bullet)/\partial K_{25}$
u_{12}	2.25× 10E-5	1.00× 10E-4	-1.81× 10E-5	-3.92× 10E-5
u_{14}	-8.42× 10E-5	-1.14× 10E-4	-7.46× 10E-6	1.21× 10E-5
u_{34}	-4.07× 10E-5	-2.65× 10E-5	-6.44× 10E-5	-6.73× 10E-5
u_{32}	-9.73× 10E-5	-2.04× 10E-6	-1.24× 10E-4	2.83× 10E-5

The derivative information can be incorporated into the calculation of elasticity of each OD travel time with respect to the capacity of a particular link. Table 4 shows that all the derivatives of each OD-pair travel time with respect to link capacities 9 and 24 are negative signs which means that by improving these two link capacities all the OD travel times can be reduced. Thus they should have high priorities to be included in the future network improvement.

Table 5 Derivatives of OD travel demand and net user benefit at initial link capacities

	$\partial(\bullet)/\partial K_8$	$\partial(\bullet)/\partial K_9$	$\partial(\bullet)/\partial K_{11}$	$\partial(\bullet)/\partial K_{13}$	$\partial(\bullet)/\partial K_{20}$	$\partial(\bullet)/\partial K_{22}$	$\partial(\bullet)/\partial K_{24}$	$\partial(\bullet)/\partial K_{25}$
q_{12}	-0.0873	0.1835	-0.0365	0.1526	-0.0376	-0.1679	0.0302	0.0657
q_{14}	-0.0574	0.0109	0.2938	0.2615	0.2451	0.3309	0.0217	-0.0353
q_{34}	0.2351	0.2057	0.1376	0.0996	0.1080	0.0705	0.1710	0.1789
q_{32}	0.0748	0.0392	-0.1361	-0.0230	0.1429	0.0030	0.1821	-0.0416
NB	0.8053	2.5673	1.0065	2.6697	2.4676	0.9082	2.3789	0.8784

Note that in Table 5 the signs of derivatives of the net user benefit are all positive indicating that the effect of the cost reduction of the previous network users is higher than that of the congestion caused by induced users from link capacity enhancements. From the Fig 3 it can be easily found that of those 8 candidate links capacity improvements of link 9, 13, 20, 24 contribute much more to the increase of the net user benefit than the other 4 links.

3.3 Determination of the Amount of Link Capacity Enhancement

Based on the results of sensitivity analysis at current link capacity four links were finally selected as the optimal alternatives. What to do next is to determine how much the four link capacities should be improved under the budget constraint.

Table 6 presents the final solutions of link capacity enhancement(Δc) and the corresponding OD travel demands with sensitivity analysis based algorithm by various upper budget limits(Budget 1: 30000, Budget 2: 40000, Budget 3: 50000).

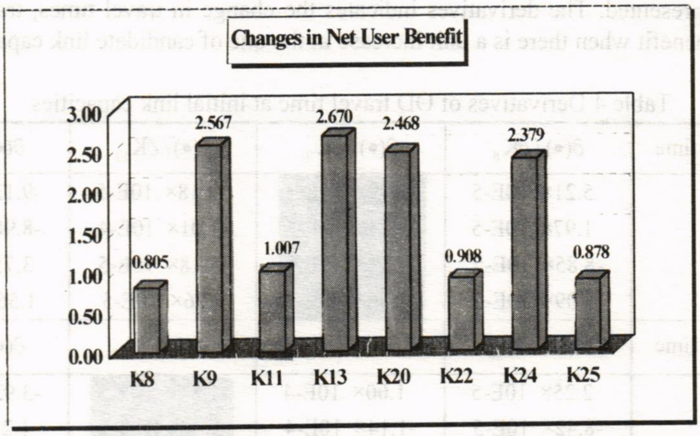

Fig. 3 Changes in Net User Benefit

The solutions are obtained with 20 major iterations and the parameter values in step size $\beta=3$, $\gamma=2$ are used.

The solution shows that as the budget limit increases, the amount of capacity improvement of link 9 and 13 is getting considerably larger compared with other links.

Fig. 4 shows the convergence process of the proposed algorithm plotting the change of upper level objective function(increase in net user benefit) vs major iterations. Although the algorithm converges very fast at first, after 10 major iterations it exhibits oscillation. This oscillation is attributed to the use of the predetermined sequences of step length method when determining the step size. It should be pointed out that more extensive numerical tests for larger networks are required to establish the efficiency of algorithms.

Table 6 Final solution of link capacity enhancements(Δc) and the corresponding OD travel demands

	Do Nothing	Budget 1	Budget 2	Budget 3
Δc_9	0	768	1,021	1,281
Δc_{13}	0	875	1,247	1,625
Δc_{20}	0	666	773	910
Δc_{24}	0	588	751	877
q_{12}	11,154	11,356	11,431	11,453
q_{14}	14,556	14,933	14,993	15,106
q_{34}	13,286	13,636	13,725	13,806
q_{32}	9,797	9,991	10,048	10,046

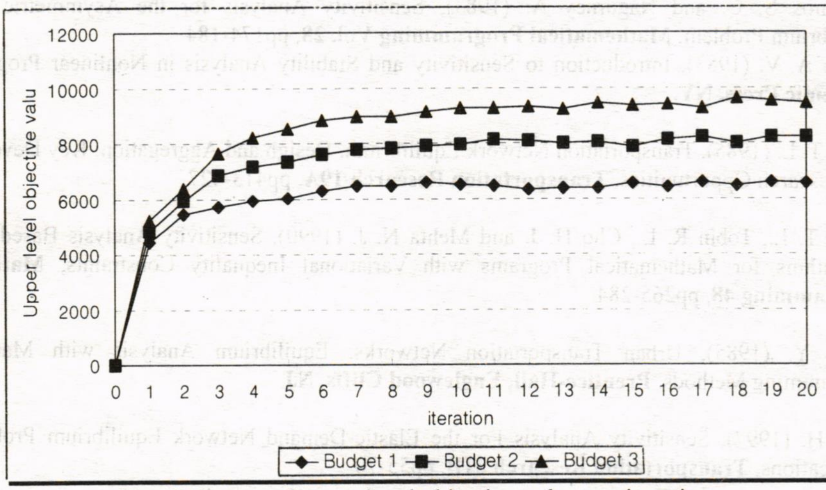

Fig. 4 Change of upper level objective values vs iterations

4. CONCLUSION

In this paper in order to overcome the limit of fixed travel demand network equilibrium a bilevel model formulation with variable travel demand was presented and the increase of net user benefit was adopted instead of minimization of total cost for the upper level problem to avoid the impractical solution. It was also shown that sensitivity analysis could be successfully applicable to equilibrium network design problem and could help select the optimal policy alternatives such as which links should be improved first. Various methods can be adopted to improve link capacity based on the network characteristics, such as widening lane widths or lateral clearance, parking controls and so on.

In the future, research will be conducted concerning about the type of demand function by the trip purpose and the calibration of its parameters reflecting the origin and destination characteristics. Furthermore efficient solution procedure applicable to large scale network design which includes substitutable improvement options is needed since enlargements of the road network rapidly increase the number of OD pairs and the dimensions of link/path and OD/path incidence matrices will grow considerably large which makes the sensitivity analysis hard to be performed.

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