STOCHASTIC TRANSIT ASSIGNMENT WITH ELASTIC DEMAND

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Abstract: This paper proposes a stochastic user equilibrium assignment model with elastic demand for congested transit networks. The stochastic effects of the passenger's behavior and overcrowded vehicle's arrivals are incorporated in the proposed model, together with the elastic transit demand. An equivalent mathematical programming problem is formulated for the stochastic transit assignment problem with elastic demand and capacity constraints. When the transit link capacity constraints are reached, it can be proven that the Lagrange multipliers of the mathematical programming problem are equivalent to the equilibrium passenger overload delays in the congested transit network. A numerical example is used to illustrate the application of the proposed model.

1. INTRODUCTION

Although the transit assignment problem has been studied in the past two decades, very little attention has been given to modeling the effect of capacity restraint on the route choice of transit passengers (Spiess and Florian 1989, De Cea and Fernández 1993, and Wu *et al.* 1994). Most of the transit assignment models being used in practice do not consider congestion effects in transit networks. These models tend to oversimplify the passenger waiting time estimation at transit stations and adopt the assumptions of unlimited line capacities and of fixed transit demand between origin and destination (OD). Recently, Lam *et al.* (1999) proposed a stochastic user equilibrium (SUE) assignments model for congested transit network, together with a solution algorithm. In Lam's model, a SUE transit assignment problem with explicit capacity constraints was considered. In the aforementioned models, however, it was assumed that the transit demand by OD pair is given and fixed, i.e. the transit demand is insensitive to the congestion of the transit network.

Concern has been given to the assumption of fixed OD demand when assessing the traffic consequences of expanding transport system or changing transit lines. It is because network improvement can have effects on the generation and distribution of travel demand. For road networks, the stochastic user equilibrium assignment model with elastic demand has been studied in recent years (Bell and Iida, 1997). In the previous related studies, the number of trips between each origin and destination (OD demand) is taken as a function of the expected minimum travel time between the OD pair, on the basis that trip-makers will choose the path with the perceived minimum travel time. However the existing assignment models with elastic demand have not yet considered the case of link capacity constraints. We cannot apply directly these assignment models to the congested transit networks.

Additionally, it was also assumed in the user equilibrium (UE) transit assignment model (Spiess and Florian 1989, De Cea and Fernández 1993, and Wu *et al.* 1994) that transit passengers have identical perceptions about the travel costs involved. This has frequently been interpreted as perfect information available to the transit passengers. The UE assumption is clearly unrealistic in practice, particularly in transit networks operating with high congestion levels. In order to study the transit passenger's behavior in response to unreliable transit services during peak hour periods, the allowance for less than perfect information is clearly helpful.

This paper extends the Lam's SUE assignment model for congested transit network to the case with elastic demand. An equivalent mathematical programming problem is formulated, where the effects of elastic OD demand and transit congestion are incorporated in the proposed model. On the other hand, the passenger overload delay at a station could be determined endogenously according to the equilibrium characteristics of the congested transit network with bottlenecks. When the transit link capacity constraints are reached, it can be proven that the Lagrange multipliers of the mathematical programming problem are equivalent to the equilibrium passenger overload delays (Lam *et al.*, 1999). Meanwhile the transit OD demand will be extended to elastic demand that is sensitive to the congestion of transit network and taken as a function of expected minimum perceived cost between origin and destination.

This paper has been organized as follows. In the next section, the network representation is defined and some useful concepts for transit network are introduced briefly. In Section 3, we will review the SUE assignment with elastic demand on road network together with some definitions. It follows in Section 4 that a SUE assignment model with elastic demand is proposed for congested transit network together with solution algorithm. In Section 5, a numerical example is given. Finally, conclusions are drawn in Section 6.

2. NETWORK REPRESENTATION AND SOME USEFUL CONCEPTS FOR TRANSIT NETWORKS

2.1 Some useful concepts

A transit network constitutes a set of stations (nodes) where passengers can board, alight or change vehicles and a set of transit lines. A transit line can be described by the frequency of the vehicles (i.e. the number of vehicles of a transit line going across a screenline in a unit of time) and the vehicle types (e.g. bus or underground train). Note that in this paper the walk links will not be distinguished from the transit lines because it may be replaced by a transit line with a zero waiting time (very high service frequency). Different transit lines may run parallel for part of their itineraries with some stations in common. A line segment is a portion of a transit line between two consecutive stations of its itinerary and is characterized with travel time and frequency.

A transit route is the feasible path that a transit passenger can follow on the transit network in order to travel between any two nodes. Generally it will be identified by a sequence of nodes, including origin node, destination node and all the intermediate nodes being the representing transfer points. The portion of a route between two consecutive transfer nodes is defined "route section" that is associated with a set of attractive lines or common lines and determined as described in De Cea and Fernández (1993)'s model. The attractive set of lines is the set of transit lines that are chosen by passengers to minimize their expected total travel time. For a congested transit network, De Cea and Fernández (1993) also extended the definition of attractive lines.

We denote a transit network by G=(N, L) with node set N, representing transit stops and link set L, representing transit network links. In this paper, only two types of link are considered:

In-vehicle links: corresponding to segments of transit lines.
Waiting links: are used to represent the linkage between alighting node/stop and boarding node/stop within a large station and/or transfer stop.

2.2 Basic notation and general assumptions

Given a transit network G(N,L), the notation used throughout this paper is given as follows.

$\bar{A}$: all attractive set of transit lines on network G;
W: set of all OD pairs.
w: an element of set W.
R_w: set of feasible routes associated with OD pair *w*.
A_i^+: the set of outgoing links at node *i*.
A_i^-: the set of incoming links at node *i*.
$\bar{A}_i^+$: corresponding to the set of all outgoing attractive lines for each destination associated with node *i* which is equal to $A_i^+ \cap \bar{A}$.
f_s: frequency of link *s*.
k_s: capacity of link *s*.
c_s: travel time or cost of link s
g_w: the elastic demand between OD pair *w,* is a function of the expected minimum perceived travel time between origin and destination.
t_s: in-vehicle time on link *s*.
u_s: waiting time of link s.
d_s: the equilibrium passenger overload delay, i.e. the time penalty that passengers will wait for the next coming vehicle or transfer to the alternative routes when they can't board the first coming vehicle because of insufficient capacity of in-vehicle links.
v_s: total passengers flow on link *s*.
h_r^w: passenger flow on route $r \in R_w$.
v: the vector of link flow.
k: the vector of link capacity.
g: the vector of OD demand.

h: the vector of route or path flow.

In this paper, the steady state assumption is adopted. This assumption will however not limit the application of the built model for the purpose of strategic planning. Obviously, a temporary saturated steady-state may exist for a short duration during the peak period when passenger flows equal limited actual capacities on a set of links, and temporary steady-state passenger queuing holds. It is also assumed that passengers do not have perfect knowledge of the timetable for the transit lines and would select the transit route which minimizes their perceived total travel time (in-vehicle plus waiting time as well as passenger overload delay).

3. THE SUE ASSIGNMENT WITH ELASTIC DEMAND

The SUE assignment proposed by Daganzo and Sheffi (1977), and Fisk (1980) is well known as a general model that consistently unifies the concept of the stochastic assignment and Wardropian equilibrium. It overcomes the shortcoming of the homogeneous user assumption in the Wardropian equilibrium, but also includes the random effect of the stochastic assignment problem on a congested road network.

Consider a road network $F=(S, E)$, where S is the intersection set and E is the set of links in a road network. Let W be the set of all OD pairs in the network and R_w be the set of routes between OD pair $w \in W$. We consider route choice behavior of a large homogeneous group of travelers with identical characteristics. Suppose each route is associated with a given actual travel time. Due to variations in perception, the path travel time is perceived differently by each traveler and thus the perceived travel time of each route is treated as a random variable. Let C_r^w represent the perceived travel time on route $r \in R_w$, which is a random variable. Also let c_r^w be the actual travel time on route $r \in R_w$. Assume that

$$C_r^w = c_r^w + \xi_r^w, \ r \in R_w, \ w \in W \tag{1}$$

where ξ_r^w is a random error term associated with the route under consideration. Furthermore, assume that $E[\xi_r^w]=0$, or $E[C_r^w]=c_r^w$ which means that the average perceived path travel time is equal to the actual travel time on the route. Furthermore, in order to take into account the effect of traffic congestion, we assume that the actual travel time for each link is a function of the flow on that link. This can be described by an increasing and strictly convex function of link flow, $t_a = t_a(v_a)$. Therefore, the actual route travel time is given by

$$c_r^w = \sum_{a \in A} t_a(v_a)\delta_{ar}^w, \quad r \in R_w, \ w \in W \tag{2}$$

where $\delta_{ar}^w=1$ if route r between OD pair w uses link a, and 0 otherwise. The probability of travelers choosing the r-th route, P_r^w, is given by:

$$P_r^w = \Pr\left(C_r^w \le C_k^w, \ \forall k \in R_w\right) \ r \in R_w, \ w \in W. \tag{3}$$

Suppose the random variable in eqn. (1) to be identically and independently distributed (i.i.d.) Gumbel variables, then the choice probabilities are specified as the logit route choice probabilities:

$$P_r^w = \frac{\exp(-\theta c_r^w)}{\sum_{k \in R_w} \exp(-\theta c_k^w)}, \ r \in R_w, \ w \in W \tag{4}$$

where the positive value of parameter θ is related to the standard deviation of the random term. This variability parameter measures the sensitivity of route choices to travel time. As $\theta \to \infty$ route choices become extremely concentrated on the least-cost route of each R_w.

SUE is defined as the state when no traveler believes that his perceived travel time can be improved unilaterally by changing routes (Sheffi, 1985).

We now review the logit-based SUE assignment model with elastic demand. The demand between an OD pair is given to be a continuous and monotonically decreasing function of the expected minimum perceived travel time between that OD pair. In other words,

$$g_w = G_w\left[\bar{S}_w(c_w)\right] \tag{5}$$

where $\bar{S}_w(c_w)$ is the expected minimum perceived travel time between OD pair $w \in W$. With a logit-based SUE assignment, the expected minimum time between an OD pair could be expressed as (see, for example, Ben-Akiva and Lerman, 1985):

$$\bar{S}_w(c_w) = E\left[\min_{r \in R_w}\{c_r^w\}\right] = -\frac{1}{\theta}\ln\sum_{r \in R_w}\exp(-\theta c_r^w) \tag{6}$$

Demand functions that are commonly encountered in the literature have an exponential form:

$$g_w = G_w(\bar{S}_w) = g_w^0\exp(-\beta\bar{S}_w) \tag{7}$$

or a linear form:

$$g_w = g_w^0 - \beta\bar{S}_w \tag{8}$$

where g_w^0 is the maximum demand between OD pair w and β is the sensitivity to the expected travel time. Now we consider the following equivalent minimization program:

(P1)

$$Min\left(\sum_{w \in W}\sum_{r \in R_w} h_r^w(\ln h_r^w - 1) - \sum_{w \in W} g_w(\ln g_w - 1)\right) + \theta\sum_{a \in A}\int_0^{v_a} t_a(x)dx - \theta\sum_{w \in W}\int_0^{g_w} G_w^{-1}(y)dy \tag{9a}$$

Subject to

$$\sum_{r \in R_w} h_r^w = g_w,\ w \in W \tag{9b}$$

$$g_w \geq 0,\ h_r^w \geq 0, r \in R_w, w \in W \tag{9c}$$

Proposition 1: The minimization program (P1) is equivalent to the logit-based SUE assignment with elastic demand defined by (5) and (6).
The proof of Proposition 1 is similar to Proposition 2 as described in Section 4.3.

4. STOCHASTIC TRANSIT ASSIGNMENT MODEL WITH ELASTIC DEMAND

4.1 Flow conservation in a congested transit network

Passenger flows on transit links that satisfy the following constraints (10), (11) and (12) are defined to be feasible. For each OD pair $w \in W$,

$$\sum_{r \in R_w} h_r^w = g_w, \quad w \in W \tag{10}$$

For each link $s \in \bar{A}$, i.e. s lies on a line of set $\bar{A}$,

$$v_s = \sum_{w \in W} \sum_{r \in R_w} a_{sr} h_r^w , \tag{11}$$

$$v_s \le k_s . \tag{12}$$

Let **A** be the link-route incidence matrix with elements a_{sr} equal 1 if link *s* lies on route *r* and equal 0 otherwise. The relationship between the passenger path flows and link flows, shown in eqn. (11), may then be expressed as

$$\mathbf{v=Ah} \tag{13}$$

Furthermore, let **B** be the route OD incidence matrix with elements b_{wr} being equal to 1 if route *r* connects OD pair *w* and 0 otherwise. The relationship between the OD flows and passenger path flows, shown in eqn. (9), may then be expressed as

$$\mathbf{g=Bh} \tag{14}$$

4.2 Actual link travel time function

The link travel time c_s consists of three components, namely: the in-vehicle time t_s, the waiting time u_s, and the equilibrium passenger overload delay d_s. Thus for each link s, $s \in \bar{A}$, the actual link time is:

$$c_s = t_s + u_s + d_s , \tag{15}$$

where t_s=0, d_s=0, if link s is a waiting link; u_s=0, if link s is an in-vehicle link. Passenger waiting at transit stations can be considered as a complex queuing process. The method of estimating passenger waiting time u_s has been described by Lam *et al.* (1999). It is defined that passenger overload delay d_s at the in-vehicle link s connecting to station i increase if $v_s > k_s$ and equal zero if $v_s \le k_s$ $(s \in \bar{A}_i^+,\ i \in N)$. It is because there will be a passenger overload delay (i.e. d_s>0) when some passengers can not get on the first arrival vehicle at the in-vehicle link s connecting to station i due to insufficient capacity of the arrival vehicle. The relationships can be expressed as:

$$\begin{cases} d_s = 0, \text{ if } v_s < k_s . \\ d_s \ge 0, \text{ if } v_s = k_s . \end{cases} \quad s \in \bar{A}_i^+ ,\ i \in N . \tag{16}$$

In a transit network with bottlenecks, only a proportion of passengers may get on the first arriving vehicle at some stations and therefore the passenger overload delay needs to be modeled carefully because of insufficient capacity on the in-vehicle links. Lam *et al.* (1999) determined endogenously the passenger overload delay according to the characteristics of the congested transit network.

4.3 The SUE assignment in a transit network

Given the transit network G= (N, L) and the corresponding attractive set of transit lines $\bar{A}$. We now consider the passenger behavior on route choice. In Section 4.2, each transit route is associated with a given actual travel time (including in-vehicle, waiting time as well as the passenger overload delay if any). In a congested transit network, due to variations in perception of passengers and the effects of other stochastic factors such as weather and incident, the path travel times are perceived differently by each passenger and thus the perceived total travel times on each route should be treated as random variables. So the SUE assignment in a transit network is more realistic in practice

Suppose T_r^w, U_r^w and D_r^w represent the passenger perceived in-vehicle time on route $r \in R_w$ respectively. They are random variables. Also let t_r^w, u_r^w and d_r^w be the actual passenger in-vehicle time, waiting time and overload delay on route $r \in R_w$. Assume that

$$T_r^w = t_r^w + \tau_r^w, \qquad r \in R_w, w \in W \tag{17}$$

$$U_r^w = u_r^w + \eta_r^w, \qquad r \in R_w, w \in W \tag{18}$$

$$D_r^w = d_r^w + \varsigma_r^w \qquad r \in R_w, w \in W \tag{19}$$

where τ_r^w, η_r^w and ς_r^w are random error terms. In addition, C_r^w is denoted as the passenger perceived total travel time on route $r \in R_w$, and C_r^w is therefore a random variable. Also let c_r^w be the passenger actual total travel time on route $r \in R_w$. Thus, we can obtain that

$$c_r^w = t_r^w + u_r^w + d_r^w$$

and

$$C_r^w = c_r^w + \xi_r^w, \ r \in R_w, w \in W \tag{20}$$

where $\xi_r^w = \tau_r^w + \eta_r^w + \varsigma_r^w$ is a random error term, which is associated with the route under consideration. Furthermore, it is assumed that $E[\xi_r^w]=0$, or $E[C_r^w]=c_r^w$; which means that the passenger perceived total travel time is equal to the actual total travel time on the route concerned.

Among the route choice models, multinomial logit and probit random utility models are commonly being used in practice. Although the assumption of identical distribution and that non-correlation in multinomial logit model is doubtful, the logit model is applicable in many cases particularly in travel demand and choice set analyses (Sheffi,1985). Alternatively, the Monte-Carlo simulation method can be used for the probit model. However, the complexity of the probit model and its considerable computation burden impede its application for large-size practical problem (Sheffi,1985). Therefore, the logit model is adopted in this paper for the SUE transit assignment problem. On the basis of the works of Bell *et al.* (1993) and Bell (1995) on alternatives to Dial's logit assignment algorithm and of Chen and Alfa (1991) on the *Method of Successive Averages (MSA),* the logit-based SUE can now be solved easily.

Definition A SUE is achieved in a congested transit network when the allocation of passengers between alternative routes conforms to the following logit model

$$\ln(h_r^w / h_{r'}^w) = -\theta(c_r^w - c_{r'}^w), \tag{21}$$

where r and r' are the alternative routes (or paths) connecting the same OD pair *w,* and $\theta>0$ is a given parameter which is used to measure the different degree of passengers' perception on the path travel time. In general, the corresponding θ value for bus network should be smaller than the one for underground transit system. As $\theta \to \infty$, the results of SUE approximate to that of deterministic user equilibrium (UE).

Consider two routes r and r' connecting the same OD pair w. The actual total travel time consists of the sum of the actual in-vehicle time, passenger waiting time on route $r \in R_w$ and the passenger overload delay time on route $r \in R_w$. Eqn.(21) suggests that $\ln(h_r^w / h_{r'}^w) = -\theta[(t_r^w - t_{r'}^w) + (u_r^w - u_{r'}^w) + (d_r^w - d_{r'}^w)]$. Obviously, if $h_r^w < k_r^w$ and $h_{r'}^w < k_{r'}^w$ then $d_r^w = d_{r'}^w = 0$. As the total demand increases, the proportionate distribution of passenger flow between the two routes remains the same until one or more links on either route are saturated. If $h_{r'}^{w} = k_{r'}^{w}$, further increase in total demand would cause congestion on route r', leading to passenger queuing. Passenger overload delay increases with the passenger queue at station and hence affects the route choice. This is the equilibrium mechanism of the logit-based assignment model proposed in this paper.

On the basis of Bell's (1993) work, we incorporate the effects of congestion and transit elastic demand in the following logit model. Consider the following problem:

$$\text{(P2)}\quad Min \sum_{w\in W}\sum_{r\in R_w} h_r^w(\ln h_r^w - 1) - \sum_{w\in W} g_w(\ln g_w - 1)$$

$$+\theta\sum_{s\in \bar{A}}(t_s + u_s)v_s - \theta \sum_{w\in W}\int_0^{g_w} G_w^{-1}(y)dy \tag{22a}$$

$$\text{s.t.}\quad g_w = \sum_{r\in R_w} h_r^w, \quad w\in W \tag{22b}$$

$$v_s = \sum_{w\in W}\sum_{r\in R_w} a_{sr}h_r^w \quad s\in \bar{A} \tag{22c}$$

$$v_s \le k_s \quad s\in \bar{A} \tag{22d}$$

$$g_w \ge 0,\ h_r^w \ge 0,\quad r\in R_w, w\in W \tag{22e}$$

Here link $s\in \bar{A}$ means that s belongs to an attractive line of $\bar{A}$.

Proposition 2: The minimization program (P2) is equivalent to the logit-based SUE assignment with elastic demand defined by (5), (6) and (21).

Proof: Substituting the constraint (22c) directly to the objective function (22a) and constraint (22d), the Lagrangian function for problem (P2) can be formulated as below:

$$L = \sum_{w\in W}\sum_{r\in R_w} h_r^w(\ln h_r^w - 1) - \sum_{w\in W} g_w(\ln g_w - 1) + \theta\sum_{s\in A}(t_s + u_s)\sum_{w\in W}\sum_{r\in R_w} a_{sr}h_r^w - \theta\sum_{w\in W}\int_0^{g_w} G_w^{-1}(y)dy$$

$$+\sum_{w\in W} l_w\Big(g_w - \sum_{r\in R_w} h_r^w\Big) + \sum_{s\in \bar{A}} m_s\Big(k_s - \sum_{w\in W}\sum_{r\in R_w} a_{sr}h_r^w\Big) \tag{23}$$

The Kuhn-Tucker conditions of problem (P2) can be given as follows:

$$\ln h_r^w + \theta(t_r^w + u_r^w) - m_r^w - l_w = 0 \quad r\in R_w, w\in W \tag{24}$$

$$-\ln g_w - \theta G_w^{-1}(g_w) + l_w = 0 \quad w\in W \tag{25}$$

$$g_w = \sum_{r\in R_w} h_r^w, \quad w\in W \tag{26}$$

$$m_s\Big(k_s - \sum_{w\in W}\sum_{r\in R_w} a_{sr}h_r^w\Big) = 0 \tag{27}$$

$$m_s \le 0 \tag{28}$$

$$\sum_{w\in W}\sum_{r\in R_w} a_{rs}h_r^w \le k_s \quad s\in \bar{A} \tag{29}$$

The form of eqns.(24) and (25) ensures that $h_r^w > 0$ and $g_w > 0$. Eqn. (24) can be rewritten as

$$\ln h_r^w = -\theta(t_r^w + u_r^w) + m_r^w + l_w,\ r\in R_w, w\in W \tag{30}$$

where l_w is the corresponding Lagrangian multiplier, $(t_r^w + u_r^w)$ is the sum of the actual in-vehicle time and the passenger waiting time on route $r \in R_w$, m_r^w is the sum of Lagrangian multipliers m_s along route $r \in R_w$.

As eqn. (26), eqn. (30) can easily be transformed to the following logit model for route choice probability:

$$h_r^w = \frac{\exp(-\theta(t_r^w + u_r^w) + m_r^w)}{\sum_{k \in R_w} \exp(-\theta(t_k^w + u_k^w) + m_k^w)} g_w \qquad r \in R_w, w \in W. \qquad (31)$$

From eqn. (31), we can see that if $m_r^w = -\theta d_r^w$, $r \in R_w$, $w \in W$, then eqn. (31) is compatible with the SUE transit assignment problem with bottlenecks. The role of m_r^w is to ensure that the capacity of route r is not exceeded. Lam *et al.* has proved that for any link, $m_s = -\theta d_s, s \in \bar{A}$, is a necessary and sufficient condition for SUE transit assignment with bottlenecks.

We now focus on the routes belonging to an specific OD pair $w \in W$. Note that $c_r^w = t_r^w + u_r^w + d_r^w$, then eqn. (24) can be rewritten as follow:

$$\ln h_r^w = -\theta c_r^w + l_w, r \in R_w, w \in W \qquad (32)$$

or

$$h_r^w = \exp(-\theta c_r^w + l_w) \qquad (33)$$

According to eqn. (26), we can obtain :

$$g_w = \sum_{r \in R_w} \exp(-\theta c_r^w + l_w) \qquad (34)$$

As eqn. (25),

$$\exp(l_w) = g_w \exp(\theta G_w^{-1}(g_w))$$

Then

$$g_w = G_w(-\frac{1}{\theta} \ln \sum_{r \in R_w} \exp(-\theta c_r^w)), \quad w \in W \qquad (35)$$

Evidently, eqn. (35) is equivalent to the demand function defined by (5) and (6). This completes the proof.

It should be noted that the objective function of (P2) can be shown to be strictly convex to the passenger path flows. The passenger path flow variable is uniquely defined and hence the passenger link flows are also uniquely determined by eqn. (11). However, the Lagrange multipliers or passenger overload delay may not be determined uniquely. On the basis of the Proposition 4 in Bell (1995), it can be concluded that the linear independence of the capacity constraints is a necessary and sufficient condition for the equilibrium passenger overload delay.

4.4 Solution algorithm for the SUE transit assignment problem with elastic demand

If the in-vehicle link capacity constraints are ignored, the problem (P2) becomes a SUE traffic assignment with elastic demand on the transit network G. There are several efficient algorithms such as the method of successive averages (Bell and Iida 1997). As the SUE transit assignment problem (P2) with elastic demand is involved with the in-vehicle link capacity constraints, some conventional SUE assignment algorithms with elastic demand cannot be applied directly. An algorithm has been proposed by Bell (1995) for solving a SUE road traffic assignment problem with queues with explicit capacity constraints. Similarly this algorithm developed by Bell (1995) can be adapted to solve the SUE transit assignment problem (P2) with elastic demand and bottlenecks.

In order to adapt the Bell's algorithm for solving the problem (P2), eqn. (31) is rewritten as below:

$$h_r^w = \exp(-\theta(t_r^w + u_r^w) + m_r^w + l_w) = \exp(-\theta(t_r^w + u_r^w)) \prod_{s \text{ in } r} M_s L_w , \tag{36}$$

where route r connects OD pair w, $L_W = \exp(l_W)$ is a factor for OD pair w, $M_s = \exp(m_s)$ is a factor for link s. Factor L_w is calculated so that eqn. (22b) holds, while factor M_s is determined so that constraints (22d) and (22e) are met. Note that each value of subscript r implies a unique value for subscript $w \in W$.

A simple procedure is proposed to solve the SUE transit assignment problem (P2) with elastic transit demand as follows:

Step 1 *(Initialization)*

$M_s^{(n)}=1$ for all links $s \in \bar{A}$, $L_w^{(n)}=1$ for all OD pairs w. Set n=1.

Step 2 *(Iteration)*

If the convergent conditions are satisfied, then stop. Otherwise, calculate the following,

For each $s \in \bar{A}$, calculate

$$h_r^w(\mathbf{L}^{(n)}, \mathbf{M}^{(n)}) = \exp(-\theta(t_r^w + u_r^w)) \prod_{s \text{ in } r} M_s^{(n)} L_w^{(n)} ,$$

$$\beta_s^{(n)} = k_s / \sum_{w \in W} \sum_{r \in R_w} a_{sr} h_r^w(\mathbf{L}^{(n)}, \mathbf{M}^{(n)}),$$

$$M_s^{(n+1)} = \min[1,\ \beta_s^{(n)}\ M_s^{(n)}].$$

For each $w \in W$, calculate

$$g_w^{(n)} = G(-\frac{1}{\theta}\ln(\sum_{r \in R_w} \exp(-\theta(t_r^w + u_r^w) \prod_{s \text{ in } r} M_s^{(n)})))$$

$$\beta_w^{(n)} = g_w^{(n)} / \sum_{r \in R_w} b_{wr} h_r^w(\mathbf{L}^{(n)}, \mathbf{M}^{(n)}),$$

$$L_w^{(n+1)} = \beta_w^{(n)}\ L_w^{(n)} .$$

Step 3 Set n=n+1, back to step 2.

Step 4 *(Output OD demand, path flows, passenger link flows, overload delays)*

For each $w \in W$, calculate

$$g_w = G(-\frac{1}{\theta}\ln(\sum_{r \in R_w} \exp(-\theta(t_r^w + u_r^w) \prod_{s \text{ in } r} M_s^{(n)})))$$

For each $r \in R_W$, calculate

$$h_r^w = h_r^w(\mathbf{L}^{(n)}, \mathbf{M}^{(n)});$$

For each $s \in \bar{A}$, calculate

$$v_s = \sum_{w \in W} \sum_{r \in R_W} a_{sr} h_r^W ,$$

$$d_s = -(\ln M_s^{(n)})/\theta .$$

Note that Step 2 in the above algorithm ensures that $M_S \le 1$ so that $m_S \le 0$, $s \in \bar{A}$. The convergence of the algorithm mentioned above will be hold (see Proposition 5 in Bell, 1995).

5. NUMERICAL EXAMPLE

Figure 1 shows an example transit network, which consists of four transit lines (L_1, L_2, L_3, L_4) serving the network and four nodes where X and Y are the transfer nodes only. The link data of the transit network in Figure 1 is given in Table 1. Let L_2^1, L_2^2 be the link L_2 from A to X and from X to Y respectively. Also, let L_3^1, L_3^2 be the link L_3 from X to Y and from Y to B respectively.

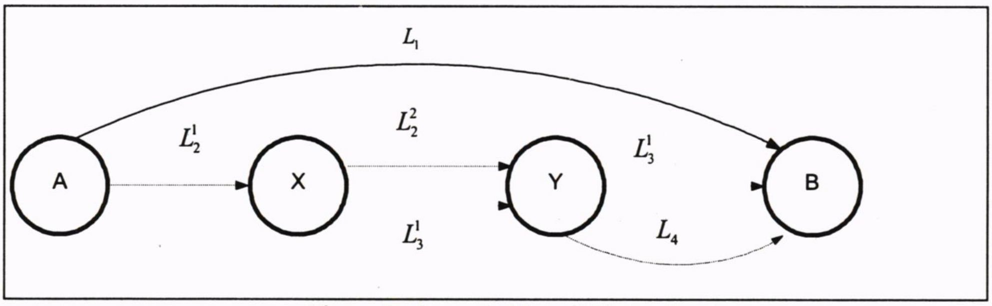

Figure 1. Example Transit Network

Table 1. Basic Link Data for the Example Transit Network

	Links					
Basic Data	L_1	L_2^1	L_2^2	L_3^1	L_3^2	L_4
f_s (veh/hr)	5	5	5	5	5	5
t_s (min)	30	12	12	10	10	10
K_s (pass/hr)	200	250	250	100	100	100

It is assumed that there is only one OD pair from origin node A to destination node B. The OD demand function is taken the exponential form as described by eqn. (7). The maximum OD demand from A to B is 400 passengers per hour (pass/hr). The results obtained by the proposed solution algorithm are shown respectively in Table 2 and Table 3 for θ=0.1, 0.5, 2.0 and 3.0. In Table 2, it can be found that the transit demand from A to B will decrease as θ increases. On the other hand, in Table 3, there are no overload delays on line L_1 for θ=0.1, while there are overload delays of 0.62, 3.56, and 3.81 minutes on line L_1 when θ=0.5, 2.0 and 3.0 respectively. Obviously, the overload delays on the express line L_1 will increase as θ increases.

In UE assignment model with elastic demand, the OD demand is a function of minimum travel time between origin and destination (Yang, 1997). From eqn. (6), if we ignore the stochastic effects of passengers' perception on path travel time, $\bar{S}_w(c_r) = E\left[\min\{c_r^w\}\right]$ in eqn (6) is exactly the minimum travel time between OD pair w of UE. In Table 2, it also can be found that the results of SUE are close to that of UE as θ increases.

Table 2. The Resultant Link Flows and OD Demands in Comparison with UE Results

	Link flows (pass/hr)						OD demands (pass/hr)
	L_1	L_2^1	L_2^2	L_3^1	L_3^2	L_4	
SUE θ = 0.1	121.7	200	100	100	100	100	321.7
SUE θ = 0.5	200	92.3	39.6	52.7	54.1	38.2	292.3
SUE θ = 2.0	200	82.2	13.8	68.4	70.1	12.1	282.2
SUE θ = 3.0	200	81.3	5.4	75.9	76.8	4.5	281.3
UE	200	80.3	0	80.3	80.3	0	280.3

Table 3. The Resultant Overload Delays for Various θ

	Link delays (min)					
	L_1	L_2^1	L_2^2	L_3^1	L_3^2	L_4
θ = 0.1	0	0	0	0.48	0.39	0.33
θ = 0.5	0.62	0	0	0	0	0
θ = 2.0	3.56	0	0	0	0	0
θ = 3.0	3.81	0	0	0	0	0

6. CONCLUSIONS

In this paper, a SUE assignment model with elastic demand is proposed for transit networks with bottlenecks and a solution algorithm is presented. The stochastic effects of the passenger's behavior and overcrowded vehicle's arrivals are incorporated in the proposed model, together with the elastic transit demand. A mathematical programming problem is formulated and equivalent to the SUE assignment problem with elastic demand in congested transit networks. When the in-vehicle link capacity constraints are reached, it can be proven that the Lagrange multipliers of the mathematical problem are equivalent to the equilibrium passenger overload delays in the transit network (Lam *et al.*, 1999).

It should be noted the proposed model is aimed to be used for long-term planning of transit network instead of on-line operation. Therefore, the steady state assumption should not limit the application of the built model in this paper. The EMME/2 and TRIPS softwares have been widely used for transit planning by many agencies in the North America, Europe and Asia. The model and algorithm presented in this paper can also be incorporated in these softwares for modeling the congestion effects over the transit system explicitly.

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