Lagrangian Relaxation for the Capacitated Single Allocation p-Hub Median Problem

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Abstract: In this paper we consider a capacitated single allocation p-hub median problem (CSA*p*HMP). We determine the location of p hubs, the allocation of non-hub nodes to hubs in the network. This problem is formulated as 0-1 integer programming model with the objective of the minimum total transportation cost and the fixed cost associated with the establishment of hubs. Optimal solutions are obtained using Gurobi optimizer for the small sized problems. Since the CSA*p*HMP is NP-hard, it is difficult to obtain optimal solution within a reasonable computational time. Therefore, a Lagrangian relaxation (LR) heuristic is developed to solve the problem. The LR performance is examined through a comparative study. The experimental results show that the proposed LR heuristic can be a viable solution method for the capacitated hub and spoke network design problem.

Keywords: *Lagrangian relaxation, p-hub median problem, single allocation, hub-and-spoke*

1. INTRODUCTION

The hub location problem (HLP) is to locate hub facilities and allocating demand nodes to hubs in order to route the flow between every origin-destination pair of nodes. Instead of providing direct links to every origin-destination pair, the hubs serve as trans-shipment points or switching points for flows between non-hub nodes. Flows departing from an origin are collected in a hub, transferred between hubs if necessary, and finally distributed to a destination node by combining with flows that are heading for the same destinations. The hub facilities consolidate flows in order to take advantage of economies of scale in transportation rate between hubs. Such a hub-and-spoke network allows many origins and destinations to be connected with fewer links than would be required with direct connections. This problem has applications in telecommunication, transportation and postal delivery systems. For example, Takano and Arai (2009) applied the *p*HMP for the Asian hub ports with Los Angeles and Rotterdam in containerized cargo transport, whereas Lin et al. (2012) solved the Chinese H&S air cargo network.

The fundamental HLP has been extended in many features, such as node allocation type, hub capacity limitation, and the number of hubs is known or unknown priori. For node allocation, each non-hub node can be allocated either to one hub (single allocation) or to multiple hubs (multiple allocation). The hub capacity could be uncapacitated or capacitated. The latter one is close to the realistic condition. If the number of hubs is pre-determined to *p*, it is called *p*-hub location problem (*p*HLP). Alumur and Kara (2008), Campbell et al. (2002), Klincewicz (1998) provided a good survey on different hub location problems. Recently, Campbell and O'Kelly (2012) provided some commentary on the present status in transportation-oriented hub location research. We refer interested readers to the paper and references therein.

In this paper, we deal with a particular variant of the *p*HLP, known as the capacitated single allocation *p*-hub median problem (CSA*p*HMP). The objective is to minimize the sum of the overall fixed cost of established *p* hubs and transportation cost in a network with *n* demand nodes and the following assumptions: the number of hubs is pre-determined (*p*), capacity limitation on total incoming and outgoing flows of candidate nodes, each non-hub node is assigned to a single hub, a discount factor for using inter-hub connection, no direct service between two non-hub nodes is allowed. The flow between every OD pair of nodes must be routed via either one or two hubs. The CSA*p*HMP is a *NP*-hard problem. Exact solution approach cannot provide solutions for large scale practical hub location problem in a reasonable computational time.

Most of the literature in hub location problem deals with uncapacitated variant of the problem, where there is not capacity limitation on hubs. One of the uncapacitated variants that has been studied is the uncapacitated single allocation hub location problem (USAHLP). O'Kelly (1987) presented a quadratic integer programming formulation for the single allocation *p*-hub median problem and proposed two heuristic methods to solve the problem. Both heuristics enumerate all possible choices of *p* hub locations. Aykin (1990) formulated the difference in the objective function if a non-hub node is assigned to different hub and defined a procedure to find the optimal allocation of non-hub to a given set of hubs. Campbell (1992) provided a linear integer programming model of the multiple allocation *p*-median problem.

Klincewicz (1991) developed several heuristics based on local improvement considering both the single and double exchange procedures and clustering of nodes. Later Klincewicz (1992) presented a tabu search (TS) and a greedy randomized adaptive search procedure (GRASP) heuristic where non-hub nodes are assigned to the nearest hub. Skorin-Kapov and Skorin-Kapov (1994) computed the results of CAB data set by tabu search and compared with the heuristics of O'Kelly (1987) and TS of Klincewicz (1992). Their results were superior but required longer computational time. Campbell (1994) produced a mixed integer linear programming (LP) formulation for four types of discrete hub location problems, the *p*-hub median problem, the uncapacitated hub location problem, *p*-hub center problems and hub covering problems, which extend the hub location problem to consider more reality situations. Skorin-Kapov *et al*. (1996) modified Campbell (1994) formulation with tighter mixed integer linear programming relaxations. They computed the result of uncapacitated multiple allocation *p*-hub location problem (UMA*p*HLP). The results were not guaranteed to obtain all integral solutions, but the objective function value is less than 1% below the optimal value which is obtained by using CPLEX.

Campbell (1996) proposed two new heuristics, MAXFLO and ALLFLO, for the single allocation *p*-hub median problem based on the multiple allocation *p*-hub median solutions. In these two heuristics, the allocations are done according to different rules but location decisions are the same. O'Kelly et al. (1996) modified Skorin-Kapov *et al*. (1996) model by assuming a symmetric flow data and further reducing the problem size. The formulation can find integer solutions most of the time. They also provided the sensitivity analysis of the solution in terms of number of hubs and hub locations to the inter-hub discount factor α. Smith *et al*. (1996) mapped the SA*p*HMP onto a modified Hopfield neural network using O'Kelly (1987) quadratic integer programming formulation. A practical postal delivery network is used to demonstrate that the quality of these Hopfield network solutions compares favorably to those obtained using both exact method and simulated annealing.

Ernst and Krishnamoorthy (1996) provided a linear integer programming formulation for uncapacitated single allocation p-hub median problem. They developed a branch and bound method to solve the problem and also demonstrated that the simulated annealing (SA) algorithm can obtain the upper bounds to improve the general branch and bound method. The SA is also showed its comparability with the TS of Skorin-Kapov and Skorin-Kapov (1994) in terms of solution quality and computational time. However, they cannot solve any problem with more than 50 nodes.

Sohn and Park (1997) provided a linear programming formulation for single allocation two-hub median problem and showed that the two-hub location problem can be solved in polynomial time. Sohn and Park (1998) presented a further reduction of Skorin-Kapov *el al*. (1996) formulation for a model with fixed hub locations when the unit transportation cost is symmetric and proportional to the distance. Later Sohn and park (2000) extended the two-hub location problem to a three-hub location problem. They transformed the three-hub location problem into a three-terminal cut problem and showed that this is a NP-hard problem.

Ernst and Krishnamoorthy (1998) proposed another branch-and-bound approach which solved shortest-path problems for each origin-destination pair to obtain lower bounds. They solved the largest single allocation problem to date with this algorithm with 100 nodes and with p = 2 and 3. Pirkul and Shcilling (1998) developed a Lagrangian relaxation method based on the Skorin-Kapov *et al*. (1996) formulation. They used subgradient optimization on the Lagrangian relaxation of the model and provided a cut constraint for one of the subproblems. In computational experiments on eighty-four standard test problems, average gaps are 0.048% and the maximum gaps are under 1%.

Ebery (2001) optimized the new proposed formulas with two or three hubs using CPLEX, and also claimed that such a formula has the potential to solve even larger problems. Abdinnour-helm (2001) discussed the solution quality by using the simulated annealing method to solve USA*p*HMP and compared with the MAXFLO, ALLFLO, and tabu search. However, their results were not as good as those obtained by Ernst and Krishnamoorthy (1996).

Pérez *et al.* (2007) presented a hybrid algorithm that merges the variable neighbourhood search (VNS) and the path-relinking (PR) paradigms. Both VNS and PR use systematic neighbourhood-based strategies to explore the feasible region and yield adequate results even with large-sized problems. Their computational results showed that the proposed hybrid algorithm constitutes an efficient alternative for solving the p-hub median problem. Kratica *et al*. (2007) constructed two genetic algorithms (Gas) for the USApHMP. The numerical experiments showed that the GAs can solve the problem with up to 200 nodes and 20 hubs. Takano and Arai (2009) presented a genetic algorithm for the hub-and-spoke problem (GAHP) for the liner shipping with shuttle service. The GAHP was first validated by CAB instances and an example of the H&S network with shuttle services with 18 ports was also analyzed. Ilić et al. (2010) proposed a general variable neighborhood search (GVNS) approach for the USA*p*HMP. The experimentation shows that GVNS outperform the comparing heuristics in terms of solution quality and computational times.

To our knowledge, the majority of research has addressed uncapacitated HLPs. There are just a few articles in the literature dealing with capacitated allocation version of HLPs. Aykin (1994) studied the capacitated hub location problem with direct links, as well as multiple allocation, and included capacity limitations and fixed costs for hubs. Two Lagrangian relaxation based heuristics are proposed. Ernst and Krishnamoorthy (1999) extended the Skorin-Kapov *et al*. (1996) formulation to the capacitated case and also proposed a mixed integer programming formulation. They proposed a simulated annealing to solve the problem. Ebery et al. (2000) presented formulations and shortest path based solution approaches for the capacitated multiple allocation hub location problem. Both CAB and AP data set were tested.

Labbé et al. (2005) studied a CSAHLP where only the operation costs associated with the flow were considered. A branch-and-cut algorithm was proposed to solve the problem up to 50 nodes. Pérez et al. (2005) proposed a GRASP-path relinking (GRASP-PR) for the CSApHMP. Computational experiments showed that GRASP-PR provided better solutions than by either GRASP or PR individually on large scale instances. Costa et al. (2008) considered the CSAHLP using bi-criteria approach. They presented two models on the second objective, the first minimizes the time to processing flows, while the second minimizes the maximum service time at the hubs.

Contreras et al. (2009) proposed a Lagrangean relaxation to obtain tight upper and lower bounds on CSAHLP. Computational experiments on benchmark instances and new generated large size instances showed that the LR obtained or improved the best known solution. Stanimirović (2010) proposed a genetic algorithm for solving the CSA*p*HMP that is to minimize the total transportation cost. The GA can reach all optimal solution for instances up to 50 nodes. García et al. (2012) provided a new 2-index integer programming formulation for the capacitated multiple allocation *p*-hub median problem (CMA*p*HMP) and solved the problem with a branch-and-cut algorithm. Some research considered fixed cost of the selected hubs (O'kelly et al., 1996; Contreras et al., 2009), while others only considered the transportation cost (Campbell, 1996; Labbé et al., 2005).

The remainder of this paper is organized as follows. In Section 2 we introduce the mathematical formulations of the CSU*p*HMP. The proposed Lagrangian relaxation heuristic and the structure of the subproblems are described in Section 3. Computational results of benchmark instances from the literature and comparisons with optimal solutions from optimization software Gurobi are provided in Section 4. We conclude with directions for further research in Section 5.

2. MATHEMATICAL FORMULATION

Consider a network of n demand nodes and p hubs must be located. Each non-hub node is allocated to a single hub. The flow between an OD (i, j) pair must be routed through either one or at most two hubs k and l. The cost of transport a unit of flow along the path i-k-l-j is computed as C_{ijkl}. The transportation cost for an OD (i, j) pair served via hubs k and m includes cost for collection from the origin i to hub k, transfer between hubs k and m, and distribution from hub m to the destination j. The rate for transfer cost between hubs is less than that for collection and distribution discount due to the economics of scale. Usually, the routing cost between two hub nodes is discounted at a rate of α to reflect the savings due to economies of scale. The CSA*p*HMP consists of locating a set of pre-determined number of hubs and assigning each node to one of the selected hubs that does not violate the hub capacity constraints and minimize total cost.

2.1 Notations

The following notations are used through out the paper.

Parameters

B_k: capacity of node k

C_{ijkl}: the transportation cost of a unit of flow from node i to node j routed via hubs k and l, ($C_{ijkl} = \lambda d_{ik} + \alpha d_{kl} + \delta d_{lj}$)

d_{ij}: the distance between nodes i and j

f_k: fixed cost of locating at node k

N: set of nodes
p: number of required facilities
w_{ij}: the flow between nodes i and j
α: the unit flow costs for transfer
λ: the unit flow costs for collection
δ: the unit flow costs for distribution
Decision variables

$$X_{ijkl} = \begin{cases} 1 & \text{if the flow from node } i \text{ to } j \text{ routed via hubs } k \text{ and } l \\ 0 & \text{otherwise} \end{cases}$$

$$Z_{ik} = \begin{cases} 1 & \text{if node } i \text{ is allocated to hub } k \\ 0 & \text{otherwise} \end{cases}$$

2.2 The Model

In this paper, we revised the mathematical formulation for the uncapacitated pHMP presented by Campbell (1996). The mixed integer formulation is as follows.

$$Min \quad \sum_{i\in N}\sum_{j\in N}\sum_{k\in N}\sum_{l\in N} w_{ij} c_{ijkl} X_{ijkl} + \sum_{k\in N} f_k Z_{kk} \tag{1}$$

$$S.T. \quad \sum_{k\in I} Z_{kk} = p \tag{2}$$

$$Z_{ik} \le Z_{kk} \qquad \forall i,k \in N \tag{3}$$

$$\sum_{k\in N}\sum_{m\in N} X_{ijkl} = 1 \qquad \forall i,j \in N \tag{4}$$

$$\sum_{l\in N} X_{ijkl} = Z_{ik} \qquad \forall i,j,k \in N \tag{5}$$

$$\sum_{k\in N} X_{ijkl} = Z_{jl} \qquad \forall i,j,l \in N \tag{6}$$

$$\sum_{i\in N}\sum_{j\in N} w_{ij} Z_{ik} \le B_k Z_{kk} \quad \forall k \in N \qquad \forall k \in N \tag{7}$$

$$Z_{ik} \in \{0,1\} \qquad \forall i,k \in N \tag{8}$$

$$X_{ijkl} \in \{0,1\} \qquad \forall i,j,k,l \in N \tag{9}$$

The objective function, equation (1), calculates the total transportation cost of flow and the fixed cost of established facilities. Constraint (2) ensures that exactly p hubs are chosen. Constraint (3) ensures that node i can be allocated to hub k only when k is selected as a hub. Constraint (4) states that every node is allocated to exactly one hub. Constraint (5) ensures that for every destination j, the total flow from origin i to destination j routed via paths using link i-k will be nonzero only if node i is allocated to hub k. Similarly, constraints (6) assures that for every origin i and every hub k, a flow through the path i-k-l-j is feasible only if j is allocated to hub l. Constraint (7) ensures that all the assigned demand to an opened facility must less than or equal to the capacity. Constraints (8) and (9) are binary integrality constraints.

3. Lagrangian Relaxation

Lagrangian relaxation (LR) is a method to solve large-scale combinatorial optimization problems. It exploits the problem structure to obtain lower bounds on the optimal solution (in minimization problem). In order to simplify the problem, we relax the constraints that link the location/assignment variables with the flow variables (Eqs. (5) and (6)). Dualizing Eqs. (5) and (6) with Lagrangian multiplier vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, then we obtain the following Lagrangian function $L(u, v)$:

$$\begin{aligned} Min \quad L(u,v) = &\sum_{k\in N} f_k Z_{kk} - \sum_{i\in N}\sum_{j\in N}\sum_{k\in N} u_{ijk} Z_{ik} - \sum_{i\in N}\sum_{j\in N}\sum_{l\in N} v_{ijm} Z_{lm} \\ &+ \sum_{i\in N}\sum_{j\in N}\sum_{k\in N}\sum_{l\in N} (w_{ij} C_{ij}^{kl} + u_{ijk} + v_{ijl}) X_{ij}^{kl} \end{aligned} \tag{10}$$

$$S.T. \quad \sum_{k\in N} Z_{kk} = p \tag{2}$$

$$Z_{ik} \le Z_{kk} \qquad \forall i,k \in N \tag{3}$$

$$\sum_{k\in N}\sum_{l\in N} X_{ij}^{kl} = 1 \qquad \forall i,j \in N \tag{4}$$

$$\sum_{i\in N}\sum_{j\in N} w_{ij} \le B_k Z_{kk} \qquad \forall k \in N \tag{7}$$

$$X_{ij}^{kl} \in \{0,1\} \qquad \forall i,j,k,l \in N \tag{8}$$

$$Z_{ik} \in \{0,1\} \qquad \forall i,k \in N \tag{9}$$

It is noted that $L(u, v)$ could be separated into two subproblems: the problem in the space of Z variables, and the problem in the space of X variables.

The subproblem in the space of Z variable:

$$\begin{aligned} Min \quad L_Z(u,v) &= \sum_{k\in N} f_k Z_{kk} - \sum_{i\in N}\sum_{j\in N}\sum_{k\in N} u_{ijk} Z_{ik} - \sum_{i\in N}\sum_{j\in N}\sum_{l\in N} v_{ijm} Z_{jl} \\ &= \sum_{k\in N} f_k Z_{kk} - \sum_{i\in N}\sum_{k\in N}\sum_{j\in N} (u_{ijk} + v_{jik}) Z_{ik} \end{aligned} \tag{11}$$

$$S.T. \quad \sum_{k\in N} Z_{kk} = p \tag{2}$$

$$Z_{ik} \le Z_{kk} \quad \forall i,k \in N \qquad \forall i,k \in N \tag{3}$$

$$\sum_{i\in N}\sum_{j\in N} w_{ij} Z_{ik} \le B_k Z_{kk} \qquad \forall k \in N \tag{7}$$

$$\sum_{k\in N} B_k Z_{kk} \ge \sum_{i\in N}\sum_{j\in N} w_{ij} \tag{12}$$

$$z_{ik} \in \{0,1\} \qquad \forall i,k \in N \tag{9}$$

In this subproblem, feasible solution do not require that the non-hub node be assigned to more than one hub. Thus, it can occur that a node is assigned to more than on hub or not assigned at all in $L_Z(u, v)$. To prevent that capacity of selected p hubs does not satisfy the total customer demand, we add constraint (12) to make sure the capacity of selected p hubs is enough. Contreras et al. (2009) solved $L_Z(u, v)$ as the knapsack problem in CHLPSA. Ours is much harder to solve therefore we use the Gurobi optimizer to solve this problem.

The subproblem in the space of X variable:

$$Min \quad L_X(u,v) = \sum_{i\in N}\sum_{j\in N}\sum_{k\in N}\sum_{m\in N} (w_{ij} C_{ij}^{km} + u_{ijk} + v_{ijm}) X_{ij}^{km} \tag{13}$$

$$S.T. \quad \sum_{k\in N}\sum_{l\in N} X_{ij}^{kl} = 1 \qquad \forall i, j \in N \qquad (4)$$

$$X_{ij}^{kl} \in \{0,1\} \qquad \forall i, j, k, l \in N \qquad (8)$$

In $L_X(u, v)$ problem we just need to find the shortest route through k and l for each (i, j) pair which has the lowest contribution of function (13).

We apply subgradient optimization for solving the $L(u, v)$. For a given vector $(\boldsymbol{u}, \boldsymbol{v})$, the Lagrangian relaxation processes are as follows. The output of the algorithm is a best lower bound LB and UB denotes a best upper bound on the optimal value of the original problem. The step size parameter ω^n is halved if lower bound has not improved in a given number e of consecutive iterations.

Step1. $\boldsymbol{u}^0 = 0$, $\boldsymbol{v}^0 = 0$, $\omega^1 = 2$, $UB = \infty$, $LB = -\infty$, $n = 1$.

Step 2. Use optimization software Gurobi to solve $L_z(u,v)$ for selecting the p locations.

Step 3. Find the assignment for non-hub nodes based on the selected p locations found in step 2 as follow.

$$X_{ij}^{kl} = \begin{cases} 1 & \min\{w_{ij}C_{ij}^{kl} + u_{ijk}^{n} + v_{ijm}^{n} \mid k,l \in N\} \\ 0 & \text{Otherwise} \end{cases} \quad ,\forall i, j \in N$$

Step 4. Compute Lagrangian objective function (L^n), if $L^n > LB$ then $LB = L^n$.

Step 5. Set $Z_{ik} = 0$, $\forall i, k\ (i \neq k)$.

Step 5.1. For each non-hub node i, find the nearest hub node ($Z_{kk} = 1$), $\min_k = \{d_{ik} \mid Z_{kk} = 1\}, \forall i \in N$.

Step 5.2 Set $Z_{ik} = 1$

Step 6. $X_{ij}^{kl} = Z_{ik}Z_{jl}, \forall i, j, k, l \in N$

Step 7. Computing Eq. (1) objective function (Obj^n), if $Obj^n < UB$ then $UB = Obj^n$

Step 8. Compute Step size using Eq. (14)

Step 9. Updating Lagrangian multipliers, $\boldsymbol{u}$ and $\boldsymbol{v}$, using Eqs. (15) and (16)

Step 10. $n = n+1$ repeat Steps 2-9 until the stopping criterion is met.

$$t^n = \frac{\omega^n(UB - L^n)}{\sum_{i\in N}\sum_{j\in N}\sum_{k\in N}(\sum_{l\in N} X_{ij}^{kl} - Z_{ik})^2 + \sum_{i\in N}\sum_{j\in N}\sum_{l\in N}(\sum_{k\in N} X_{ij}^{kl} - Z_{jl})^2} \qquad (14)$$

$$u_{ijk}^{n+1} = u_{ijk}^{n} + t^n(\sum_{l\in N} X_{ij}^{kl} - Z_{ik}) \qquad (15)$$

$$v_{ijl}^{n+1} = v_{ijl}^{n} + t^n(\sum_{k\in N} X_{ij}^{kl} - Z_{jl}) \qquad (16)$$

For stopping criterion in Lagrangian relation heuristic, in our experiment we implement 3 stopping criterion, the algorithm terminates when one of the following condition is met.

1. The given maximum number of iterations $Iter_{\max}$ is reached.
2. The step size parameter ω^n is less than a threshold value.
3. The lower bound equals the best upper bound or is close to upper bound below a threshold value ($|UB - L^n| < \varepsilon$).

4. COMPUTATIONAL RESULTS

In this section the computational results of the proposed Lagrangian relaxation (LR) and comparison with the results obtained by using Gurobi 4.5.2 optimizer are given. All tests were carried out on the Intel Core2 Duo 3.0GHz CPU with 2 GB RAM, running under Windows 7 operations systems. The LR was coded in Microsoft Visual Studio 2010 C++ and tested on four sets of OR-Library (http://people.brunel.ac.uk/~mastjjb/jeb/info.html) instances taken from Beasley (1996). The computational time for Gurobi optimizer is set for 2 hours.

The small size AP instances up to fifty nodes from Ernst and Krishnamoorhy (1996), is derived from a study of the Australian postal (AP) delivery system. The data set contains four instances of each of the sized n = 10, 20, 25, 40, and 50. These instances include capacities and fixed costs on nodes. Ernst and Krishnamoorhy (1999) used a combination of two types of fixed cost, tight (T) and loose (L), and two types of capacities, tight (T) and loose (LT) for each problem size. Instances with fixed costs of type T have higher fixed costs for nodes with large flows. This makes it more difficult to solve. For every problem size the four instances correspond to one of the four possible combinations, LL, LT, TL, and TT. The number of hubs p in tested instances is between 2 and 5. The instances for $n > 25$ in LT and TT must have at least 3 hubs. Thus, there are 20 and 18 instances for loose and tight capacity type problem, respectively. The cost parameters are $\lambda = 3$, $\alpha = 0.75$, and $\delta = 2$.

After a preliminary test, we set the following parameter values: e = 25, $\varepsilon = 0.001$, and $Iter_{\max}$ = 1000. Tables 1 to 4 show the results of Gurobi optimizer and our Lagrangian relaxation (LR) approach for all instances. Columns 1 and 2 are number of nodes n and number of hubs p. The optimal solutions and computational time obtained by Gurobi optimizer are listed in columns 3 and 4. Columns under LR indicate the best upper bound and computation time by our LR. The Gurobi can only solve the instance up to 25 nodes ($n \le 25$). For the instances with more than 25 nodes, the Gurobi optimizer cannot find the optimal solution within 2 hours (marked as a dash "-"). For instances with $n > 25$, we present the results from Ernst and Krishnamoorhy (1999) for the CSAHLP for comparison. The solution quality is evaluated as a percentage gap with respect to the optimal solution obtained by Gurobi as eq. (17).

$$gap = \frac{\text{LR solution - Opt.}}{\text{Opt.}} \times 100\% \tag{17}$$

Table 1 provides the results for the LL instances. We observe that our LR can obtain the optimal solutions for all instances except 2 for $n \le 25$. For the instances with $n > 25$, LR can obtain the same results as those from Ernst and Krishnamoorhy (1999). The average gap of those instances with available optimal solution is only 0.29%. The computational time decreases as the number of hubs increases.

Table 2 presents the results for the LT instances. As can be seen, the number of hubs must be at least 3 for instances with $n > 25$ in tight capacity. Our LR cannot find 2 optimal solutions for instances with $n \le 25$. The LR does not show good solution quality for instances with $n > 25$ as those for loose type instances in Table 1. The average gap is 1.39% which is much higher than the loose capacity instances in Table 1.

Table 3 provides the results for the TL instances. We observe that our LR can obtain the optimal solutions for all instances except 2 for $n \le 25$. For the instances with $n > 25$, LR can obtain the same results as those from Ernst and Krishnamoorhy (1999). The average gap of those instances with available optimal solution is only 0.04%.

Table 1. The results for the LL instances

N	p	Gurobi		LR	
		Opt.	time	gap	Time
10	2	230008.5	1.14	0.00	25.28
	3	224250.1	1.21	0.00	2.32
	4	229172.6	1.32	0.00	2.31
	5	239292.3	1.21	0.00	1.28
20	2	234691.0	18.13	0.00	79.23
	3	239444.2	20.56	0.00	35.54
	4	251939.7	20.91	0.00	38.80
	5	266745.2	19.00	0.00	15.18
25	2	238978.0	87.19	0.00	190.37
	3	242437.2	88.64	0.55	132.29
	4	252716.6	123.20	0.00	64.91
	5	263518.3	59.97	2.87	74.68
40	2	241955.71[a]	-[c]	0.00	18.97
	3	-[b]	-	-	17.10
	4	-	-	-	326.07
	5	-	-	-	22.36
50	2	238520.59	-	0.00	564.40
	3	-	-	-	530.10
	4	-	-	-	34.06
	5	-	-	-	37.78
Average				0.29	110.65

[a]: Solution provided by Ernst and Krishnamoorthy (1999).
[b]: Gurobi cannot solve the instance within 2 hours.
[c]: The computational time is not available.

Table 4 presents the results for the TT instances. As can be seen, the number of hubs must be at least 3 for instances with $n > 25$ in tight capacity. Our LR cannot find 4 optimal solutions for instances with $n \leq 25$. The LR does not show good solution quality for instances with $n > 25$ as those loose type instances in Table 1. The average gap for all instances is 2.14%.

Note that the LR is able to obtain optimal solutions in most of loose capacity type instances. However, the performance of our LR algorithm does not provide good solutions in tight capacity instances. The computational time is also not fast as we expected. The reason might be that we run the subproblems with Gurobi optimizer to optimality. It could occur that Gurobi spend long time to find the optimal solution. If we use other heuristic approach, the computational time could be reduced.

5. CONCLUSION

In this paper, we consider the capacitated single allocation p-hub median problem

(CSApHMP). The number of locations is not known priori, and the amount of flow collected in the hub is limited. We propose a Lagrangian relaxation that decomposed the problem into smaller subproblems that can be solved efficiently. Four sets of benchmark instances from AP hub data set are tested for our Lagrangian relaxation heuristic. The results are also compared with the Gurobi optimizer.

Our LR algorithm is able to obtain good solutions for instances $n \leq 25$. However, the performance is not as good for $n > 25$ instances. The present study shows that the LR is an effective method to solve CSA*p*HMP, the multiple allocation of *p*-hub median problem (CMA*p*HMP) that considers a more relaxed situation could be further discussed. Future research can apply the LR algorithm to solve the CMA*p*HMP and for the real world problem. We can also add the local search to the solution found by LR to improve the solution quality, especially for the tight capacity instances. Other research direction might to develop metaheuristic algorithms, such as GRASP and ACO, to solve the problem.

Table 2. The results for LT instances

n	*p*	Gurobi		LR	
		Opt.	time	gap	Time
10	2	256048.6	1.17	0.00	46.87
	3	252973.6	1.05	0.00	137.65
	4	250992.3	1.03	0.00	22.91
	5	261451.2	1.01	0.00	38.59
20	2	253517.4	639.65	2.99	316.97
	3	257247.7	2079.15	3.18	18.39
	4	260678.9	269.24	0.00	280.05
	5	274975.4	291.50	0.00	217.75
25	3	276372.5	751.32	0.00	1494.30
	4	278235.0	271.99	0.00	1862.89
	5	284952.6	73.54	0.00	377.67
40	3	272218.32 [a]	- [c]	5.36	289.24
	4	- [b]	-	-	28.38
	5	-	-	-	303.53
50	3	-	-	-	665.93
	4	272897.49	-	6.30	49.98
	5	-	-	-	41.33
Average				1.37	364.26

[a]: Solution provided by Ernst and Krishnamoorthy (1999).
[b]: Gurobi cannot solve the instance within 2 hours.
[c]: The computational time is not available.

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Table 3. Test results for TL instances.

n	p	Gurobi		LR	
		Opt.	time	gap	Time
10	2	264544.0	1.08	0.00	12.59
	3	263399.9	1.22	0.00	2.44
	4	269074.3	1.06	0.00	0.72
	5	281327.3	1.09	0.00	0.30
20	2	271128.2	17.19	0.00	54.09
	3	281304.8	18.61	0.43	29.51
	4	295223.9	18.86	0.00	22.98
	5	310122.8	16.85	0.00	3.50
25	2	310317.6	61.15	0.00	113.42
	3	328092.6	215.45	0.00	90.56
	4	347416.8	122.83	0.00	74.59
	5	368288.6	46.82	0.07	31.63
40	2	298919.01[a]	-[c]	0.00	231.11
	3	-[b]	-	-	179.23
	4	-	-	-	116.99
	5	-	-	-	133.75
50	2	319015.77	-	0.00	381.79
	3	-	-	-	330.78
	4	-	-	-	399.26
	5	-	-	-	344.76
Average				0.04	127.70

[a]: Solution provided by Ernst and Krishnamoorthy (1999).
[b]: Gurobi cannot solve the instance within 2 hours.
[c]: The computational time is not available.

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Table 4. Test results for TT instances

n	*p*	Gurobi		LR	
		Opt.	Time	gap	Time
10	2	264544.0	1.01	0.00	2.22
	3	2633 99.9	1.12	0.00	2.25
	4	269074.3	1.19	0.00	0.48
	5	281327.3	1.06	0.00	0.28
20	2	329068.6	1849.93	9.80	54.97
	3	296035.4	24.96	0.00	70.67
	4	306296.3	14.88	0.00	9.25
	5	325568.5	14.57	0.58	10.97
25	3	348369.1	442.95	1.51	99.59
	4	369576.8	1364.41	0.00	67.05
	5	391996.5	357.94	1.16	38.47
40	3	354874.10[a]	-[c]	4.66	190.96
	4	-[b]	-	-	12.93
	5	-	-	-	147.19
50	3	-	-	-	315.37
	4	417440.99	-	10.11	28.55
	5	-	-	-	24.09
Average				2.14	63.25

[a]: Solution provided by Ernst and Krishnamoorthy (1999).
[b]: Gurobi cannot solve the instance within 2 hours.
[c]: The computational time is not available.

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