The Reliability-Based Stochastic Transit Assignment Problem with Elastic Demand

Y. JIANG[a], W.Y. SZETO[b], T.M. NG[c], SIN C. HO[d]

[a,b,c] *Department of Civil Engineering, The University of Hong Kong, Hong Kong*
[d] *Department of Economics and Business, Aarhus University, Denmark*
[a] *E-mail: jiangyu@hku.hk*
[b] *E-mail: ceszeto@hku.hk*
[c] *E-mail: tszmanng@hku.hk*
[d] *E-mail: sinch@asb.dk*

Abstract: This paper examines the reliability-based stochastic transit assignment problem with elastic demand. A Variational Inequality (VI) model for this problem is developed. The VI model considers capacity, waiting time and in-vehicle travel time as stochastic variables, and includes Spiess and Florian's (1989) and de Cea and Fernández's (1993) models as special cases. A reliability-based stochastic user equilibrium condition is defined to capture the route choice behavior of passengers. To illustrate the properties of the VI model, numerical studies were conducted on de Cea and Fernández's (1993) network. The studies also show that Spiess and Florian's and de Cea and Fernández's models can overestimate the system performance substantially.

Keywords: Reliability-Based Stochastic User Equilibrium, Elastic Demand, Transit Assignment, Variational Inequality, Effective Travel Cost

1. INTRODUCTION

Transit assignment received much attention in the past. Its models could be used for planning and managing transit services. Existing transit assignment models can be broadly classified into two types: schedule-based and frequency-based. Schedule-based models (e.g., Tong and Wong, 1998; Nguyen *et al.*, 2001; Nuzzolo *et al.*, 2001; Poon *et al.*, 2004; Hamdouch and Lawphongpanich, 2008; Sumalee *et al.*, 2009; Hamdouch *et al.*, 2011) generally use detailed departure or arrival times of each transit vehicle in making assignment decisions and determine the patronage of transit lines over time. The models are suitable for transit line operation control where the departure times of transit vehicles at each stop are the key decisions to be made. Frequency-based models (e.g., Spiess and Florian, 1989; de Cea and Fernández, 1993; Lam *et al.*, 1999, 2002; Cepeda *et al.*, 2006; Teklu, 2008; Szeto *et al.*, 2011a, b) ignore the detailed departure times and often allocate passenger demands to the set of attractive transit lines based on service frequency. The models are more computationally efficient and can handle larger transit networks. Hence, they are more suitable for designing route and frequency in large, realistic transit networks.

Although many frequency-based models were developed, four issues are worth mentioning. First, most of the existing frequency-based models (e.g., de Cea and Fernández, 1993; Cominetti and Correa, 2001; Cepeda *et al.*, 2006) only consider deterministic networks. In fact, due to supply side uncertainties such as capacity uncertainties, travel times including waiting time and in-vehicle travel time are stochastic. Hence, it is essential to capture the uncertainties into transit assignment models to have a more accurate analysis for transit planning and management purposes. Second, the behavioral choice principle of passengers in most of the frequency-based models does not consider the variability of trip time (e.g., de Cea and Fernández, 1993; Lam *et al.*, 1999; Lam *et al.*, 2002). However, empirical studies show that travel time variability indeed affects the choice behavior of passengers (Abdel-Aty *et al.*, 1997 and Jackson and Jucker, 1982). It is important to capture this variability into transit assignment models. Although some existing studies (Yang *et al.*, 2006; Li *et al.*, 2008, 2009; Chen *et al*., 2011; Sumalee *et al.*, 2011; Fu *et al.*, 2012) propose stochastic models to incorporate the uncertainties of travel time, the random variables are often assumed to follow certain distributions. Third, most of existing frequency-based models (e.g., Spiess and Florian, 1989; de Cea and Fernández, 1993; Szeto *et al*., 2011a, 2011b) assume that passengers have perfect information about the transit network status. However, in reality, they may not have as the real time information may not be given at transit stops and on the web. It is imperative to capture their choice behavior in the transit assignment models. Fourth, most studies (e.g., Spiess and Florian, 1989; de Cea and Fernández, 1993) ignore demand elasticity. Their models cannot be used to study the transit pattern for social and recreational trips where the demand is highly elastic.

To address the above issues, this paper proposes a reliability-based stochastic transit assignment model with elastic demand. Compared with most of the existing stochastic transit assignment models, the contributions of this paper include:

- Considering the stochasticity of waiting time, link capacity, and in-vehicle travel time in the model;
- Capturing the risk-averse behavior and perception error of passengers simultaneously in the proposed model;
- Incorporating transit demand elasticity into the model through an elastic demand function; and,
- Formulating the proposed transit assignment problem as a Variational Inequality problem, which includes Spiess and Florian (1989) and de Cea and Fernández (1993) models as special cases. No normal distribution assumption is required in the formulation. In fact, the formulation is distribution-free.

The remainder of the paper is organized as follows: Section 2 introduces the general representation of transit networks, terminologies, and the assumptions that are relevant to our work. Section 3 reviews the effective frequency concept. Section 4 elucidates the various components of the cost function. While Section 5 describes the idea of effective travel cost, Section 6 postulates the RSUE condition and a VI formulation for the stochastic user equilibrium transit assignment problem. Section 7 discusses the computational results. Finally, Section 8 provides concluding remarks and identifies directions for future research.

2. NETWORK REPRESENTATION, TERMINOLOGIES, AND ASSUMPTIONS

As in de Cea and Fernández (1993) and Lam *et al*. (1999, 2002), a transit network generally consists of a set of transit lines and stations (nodes) where passengers can board, alight or change vehicles. A transit line can be described by the frequency of the vehicles (i.e., the number of vehicles of a transit line going across a screenline in a unit of time) and the vehicle types (e.g., bus or underground train). Note that in this paper the walk links will not be distinguished from the transit lines because it may be replaced by a transit line with a zero waiting time (a very high service frequency). Different transit lines may run parallel for part of their itineraries with some stations in common. A line segment is a portion of any transit line between two consecutive stations of its itinerary and is characterized by a travel time and a frequency. A transit route is any path that a transit passenger can follow on the transit network in order to travel between an origin and a destination. Generally, the route will be identified by a sequence of nodes, the first node being the origin of the trip, the final node being the destination and all the intermediate nodes being the transfer points. The portion of a route between two consecutive transfer nodes is called route section or link, which is associated with a set of attractive lines or common lines. Without loss of generality, a transit network can also be represented by a set of nodes and route sections.

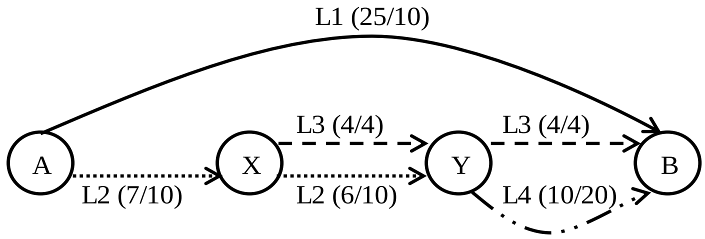


Figure 1a. Line-node representation

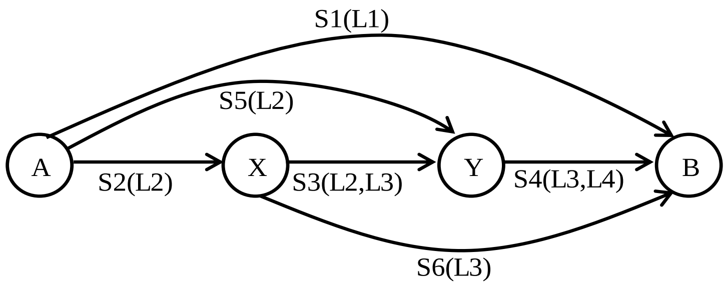


Figure 1b. Route-section representation

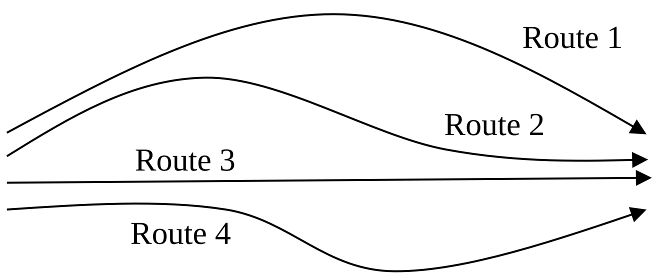


Figure 1c. Four transit routes between A and B

Figure 1. Two transit network represenations for an OD pair with four paths

For illustrative purposes, we adopt the network in de Cea and Fernández (1993) as an example. Figure 1 represents a transit network in two ways, namely line-node and route section representations. In this figure, L1 stands for line 1 and S3(L2,L3) stands for route section 3 where Lines 2 and 3 are on this route section. Other notations in the figure are defined similarly. In the figure, there is one origin-destination (OD) pair A-B and two transfer stations X and Y. OD pair A-B is connected by four different routes. The four routes are formed by four different lines, each with its travel time and frequency. For example, (25/10) on the line 1 going from A to B, denotes a travel time of 25 minutes and frequency of 10 buses/hour. At node 1, passengers need to decide to get into either L1 or L2 but not both, because they depart from different bus terminals.

We assume that a passenger waiting at a transfer node only considers an attractive set of lines before boarding. The travel demand between each OD is assumed to be elastic and passenger arrivals at bus stops are assumed to be random. We also assume that the passenger selects the transit route that minimizes his/her perceived effective travel cost discussed later. Time and cost are used interchangeably throughout this paper assuming the value of time is equal to 1. Stochastic vehicle headways with the same distribution function are assumed for vehicles servicing different lines. However, the difference in vehicle headway traversing different lines can be achieved by varying the parameters of the distribution function. We also assume that the capacity of the transit system is limited by the capacities of transit vehicles.

3. EFFECTIVE FREQUENCY

To model the effect of in-vehicle congestion in a transit network, we adopt a similar idea of effective frequency introduced by de Cea and Fernández (1993). In a transit network constrained by its capacity, there is a positive probability that a transit vehicle arriving at a stop is full. Hence, passengers have to wait for the next transit vehicle and this causes the frequency of the line at that particular stop to be effectively reduced from the passengers' point of view. This reduced line frequency is called effective frequency. In an ideal case, when there is no congestion, the effective frequency will be equal to its line frequency. Mathematically, the effective frequency can be expressed as:

$$f_s'^l = \frac{\alpha}{\dfrac{\alpha}{f^l} + \varphi_s^l}, \ \forall l \in A_s, \forall s \in \mathcal{S}, \tag{1}$$

where $f_s'^l$ is the effective frequency of line l on route section s . f^l is the frequency of line l. α is a positive parameter used to model the effect of different perceptions of waiting time and headway distributions. φ_s^l is the additional (mean) waiting time for line l at stop $i(s)$, the origin node of route section s, due to in-vehicle congestion. A_s is the set of attractive lines associated with route section s. $\mathcal{S}$ is the set of route sections.

In this paper, the additional waiting time for line l is expressed as:

$$\varphi_s^l = \beta^l \left(\frac{\bar{v}_{il}}{K^l} \right)^m, \forall l \in A_s, \forall s \in \mathcal{S}, \tag{2}$$

where K^l is the capacity of line l. $\bar{v}_{il}$ is the number of passengers per hour boarding line l before the origin node $i(s)$ of route section s and alighting after node $i(s)$. β^l and m are positive calibration parameters. Eq. (2) can be interpreted as follows: The fraction $\frac{\bar{v}_{il}}{K^l}$ is interpreted as the occupancy rate, which is a measure of in-vehicle congestion. When the occupancy rate increases, the additional waiting time increases. Moreover, for a given occupancy rate, larger values of m and β^l mean that more passengers are willing to wait at the bus stop for the next arriving vehicle, leading to higher additional waiting time.

The capacity K^l of line l is given by

$$K^l = f^l k, \ \forall l, \tag{3}$$

where k is the capacity of a transit vehicle and is assumed to be constant for all the vehicles servicing different routes for simplicity, although there is no conceptual difficulty to generalize to the situation that different routes have different vehicle capacities.

4. INDIVIDUAL COST COMPONENTS

The proposed cost function in this paper captures the variabilities of congestion and capacity. This cost function is developed based on the concept of effective travel cost discussed later and relate to the mean and variance of individual route section costs. This section describes these individual components.

The cost on route section s, C_s, is described by three random variables:

$$C_s = T_s + X_s + \varphi_s, \quad \forall s \in \mathcal{S}, \tag{4}$$

where T_s is the in-vehicle travel time on route section s. X_s is the waiting time for the first arrived vehicle on route section s that is not full. φ_s is the additional waiting time on route section s due to insufficient capacity.

4.1 In-vehicle Travel Time

Let T_s^l, the in-vehicle travel time for line l on route section s, be a random variable. Then, the in-vehicle travel time on route section s can be found using the relationship

$$T_s = \sum_{l \in A_s} w_s^l T_s^l, \quad \forall s \in \mathcal{S}, \text{ and} \tag{5}$$

$$w_s^l = \frac{f_s'^l}{\sum_{l'' \in A_s} f_s'^{l''}}, \quad \forall s \in \mathcal{S}, l \in A_s \tag{6}$$

where w_s^l is the weight based on the effective frequencies of lines, $l \in A_s$, $s \in \mathcal{S}$. Effectively, Eq. (5) calculates the weighted average of in-vehicle travel times. The expected in-vehicle travel time can be obtained by taking expectation on both sides of Eq. (5):

$$E[T_s] = \sum_{l \in A_s} w_s^l E[T_s^l], \quad \forall s \in \mathcal{S}. \tag{7}$$

Assume the in-vehicle travel times of different lines are independent. The variance of in-vehicle travel time can then be found by

$$Var[T_s] = \sum_{l \in A_s} \left(w_s^l\right)^2 Var[T_s^l], \quad \forall s \in \mathcal{S}. \tag{8}$$

In practice, in-vehicle travel times between different line sections and between different lines sharing the same route section are not likely to be independent. When these in-vehicle travel times are highly correlated, covariance terms must be added to Eq. (8) to improve the accuracy of modeling.

4.2 Waiting Time for the First Arrived Vehicle

The waiting time distribution for the arrival of the first vehicle that is not full can be derived from the headway distribution of transit vehicles as discussed in Spiess and Florian (1989) but here we incorporate the concept of effective frequency in determining the mean and variance of waiting time for the first arrived vehicles. Assuming that passengers arrive at bus stops randomly, the waiting time distribution for line *l* on route section s can be given by

$$g_s^l(x) = \frac{[1 - H_s^l(x)]}{\int_0^\infty [1 - H_s^l(t)]dt}, \ \forall l \in A_s, \forall s \in \mathcal{S}, \tag{9}$$

where $H_s^l(x)$ is the cumulative distribution of the interarrival times (or headways). By definition, the cumulative distribution function of waiting time for line *l* on route section *s*, denoted by $G_s^l(x)$, can then be obtained as

$$G_s^l(x) = P\{X_s^l \le x\} = \int_0^x g_s^l(t)dt, \ \forall l \in A_s, \forall s \in \mathcal{S}, \tag{10}$$

where X_s^l is the waiting time for line $l \in A_s$.

Using Eqs. (9) and (10), we can determine the means and variances of waiting time for a particular line *l* on route section *s* and those for route section *s* based on the assumed distribution of vehicle headway. Particular, when the headway for line l on route section *s* is exponentially distributed with mean $\alpha / f_s'^l$. Hence, we have

$$H_s^l(x) = 1 - e^{-\frac{f_s'^l}{\alpha}x}, \ \forall l \in A_s, \forall s \in \mathcal{S}. \tag{11}$$

Substituting Eq. (11) into Eq. (9) and then substituting the resulting expression into Eq.(10), we get

$$G_s^l(x) = 1 - e^{-\frac{f_s'^l}{\alpha}x}, \ \forall l \in A_s, \forall s \in \mathcal{S}, \tag{12}$$

which means that the waiting time of line *l* on route section *s* is exponentially distributed with mean $\alpha / f_s'^l$.

The mean and variance of waiting time on route section *s* can be deduced from the cumulative distribution function (12) by calculating the first and second moments of the waiting time X_s, as shown below:

$$E[X_s] = \int_0^\infty \{1 - F_{X_s}(t)\}dt, \ \forall s \in \mathcal{S}, \text{ and} \tag{13}$$

$$E[X_s^{\,2}] = 2\int_0^\infty t\{1 - F_{X_s}(t)\}dt, \ \forall s \in \mathcal{S}. \tag{14}$$

Assuming that the waiting time on each line *l* of route section *s* to be independent of each other, the brace terms in Eqs. (13) and (14) can be expressed as

$$1 - F_{X_s}(x) = P(X_s \geq x) = \prod_{l \in A_s} P(X_s^l \geq x) = \prod_{l \in A_s} \{1 - G_s^l(x)\}, \ \forall s \in \mathcal{S}. \tag{15}$$

Then, the first and second moments can be simplified by putting Eqs. (12) and (15) into both Eqs. (13) and (14) as follows:

$$E[X_s] = \int_0^\infty \prod_{l \in A_s} \{1 - G_s^l(t)\} dt = \frac{\alpha}{\sum_{l \in A_s} f_s'^l}, \ \forall s \in \mathcal{S}, \text{ and} \tag{16}$$

$$E[X_s^2] = 2\int_0^\infty t \prod_{l \in A_s} \{1 - G_s^l(t)\} dt = \frac{2\alpha^2}{(\sum_{l \in A_s} f_s'^l)^2}, \ \forall s \in \mathcal{S}. \tag{17}$$

Since the variance of X_s can be determined by

$$Var[X_s] = E[X_s^2] - (E[X_s])^2, \ \forall s \in \mathcal{S}, \tag{18}$$

we can substitute Eqs. (16) and (17) into Eq. (18) to get

$$Var(X_s) = \frac{\alpha^2}{(\sum_{l \in A_s} f_s'^l)^2}, \ \forall s \in \mathcal{S}. \tag{19}$$

4.3 Congestion Cost

The congestion cost of section *s* depends on the flow or the number of passengers per hour on route section *s*, V_s, as well as the flow on the sections competing with section *s*, $\bar{V}_s$. V_s can be obtained once all the route flows on section *s* are known:

$$V_s = \sum_{w \in \mathcal{W}} \sum_{r \in \mathcal{R}_w} b_{sr} y_r^w, \quad \forall s \in \mathcal{S}, \tag{20}$$

where y_r^w is the flow on route *r* between OD pair *w*. $\mathcal{W}$ is the set of OD pairs. $\mathcal{R}_w$ is the set of routes between OD pair *w*. b_{sr} is the route-section route incidence indicator, which equals 1 if route section *s* is a part of route r, and equals 0 otherwise.

The flow on the sections competing with section *s*, $\bar{V}_s$, is made up of two groups of passenger flows, namely group-1 and 2 flows. Group-1 flow is the number of passengers per hour boarding at node $i(s)$ and those who will not transfer to other lines and finish their trips at the destination node of route section *s*. Group-2 flow is the number of passengers per hour boarding the lines belonging to route section *s* before $i(s)$ and alighting after $i(s)$. Essentially, $\bar{V}_s$ represents the flows that compete with V_s for the capacities of the same set of attractive lines. The difference lies in that V_s represents the flow boarding at the origin of section *s* and alighting at the end of section *s*. $\bar{V}_s$ represents the flow either boarding before the origin of section *s* (i.e., group-1 flow) or alighting after section *s* (i.e., group-2 flow) using the lines belonging to the attractive set of lines on section *s*. The flow $\bar{V}_s$ excludes the flow on route section *s*, V_s. Mathematically, $\bar{V}_s$ is given as follows:

$$\bar{V}_s = \sum_{l \in A_s} \sum_{\bar{s} \in S_{ls}^+} v_{\bar{s}}^l + \sum_{l \in A_s} \sum_{\bar{s} \in \bar{S}_{ls}} v_{\bar{s}}^l, \ \forall s \in \mathcal{S}, \tag{21}$$

where v_s^l is the number of passengers per hour on line l in route section s. S_{ls}^+ is the set of route sections going out from node $i(s)$ and containing line l but excludes route section s. $\overline{S}_{ls}$ is the set of route sections containing line l with their origin nodes before $i(s)$ and their destination nodes after $i(s)$. Assuming that the passengers board on the first arrived transit vehicles, v_s^l in can be found by

$$v_s^l = \frac{f_s'^l}{\sum_{j \in A_s} f_s'^j} V_s, \quad \forall l \in A_s, \forall s \in \mathcal{S}. \tag{22}$$

The mean and variance of congestion cost is derived from the proposed congestion cost function, which is more general than the one proposed by de Cea and Fernández (1993). The congestion cost function for route section s is expressed as

$$\varphi_s = \beta_s \left(\frac{V_s + \overline{V}_s}{K_s} \right)^n, \forall s \in \mathcal{S}, \tag{23}$$

where β_s and n are calibration parameters. K_s is the capacity of route section s. V^s is the flow or number of passengers per hour on route section s.

The route section capacity K_s is defined as:

$$K_s = \frac{\gamma k}{h_s}, \ \forall s \in \mathcal{S}, \tag{24}$$

where γ is a conversion factor, and h_s is the headway of transit vehicles on route section s. k refers to the capacity of a single vehicle. If the unit for headway is minutes and that for the capacity of a line is passengers per hour, then γ = 60 min/hr.

Since headway is a random variable, the capacity is also a random variable according to Eq. (24) and hence the congestion cost is also a random variable according to Eq. (23). Substituting Eq. (24) into Eq. (23), and taking expectation and variance on both sides of the resulting expression, we get

$$E\left[\varphi_s\right] = \beta_s \left(\frac{V_s + \overline{V}_s}{\gamma k} \right)^n E[(h_s)^n], \ \forall s \in \mathcal{S}, \text{ and} \tag{25}$$

$$Var\left[\varphi_s\right] = \beta_s^2 \left(\frac{V_s + \overline{V}_s}{\gamma k} \right)^{2n} Var[(h_s)^n], \ \forall s \in \mathcal{S}, \text{ respectively.} \tag{26}$$

Since $h_s^l \sim Exp\left(\alpha / f_s'^l\right), \forall s$, according to the property of superposition of Poisson processes, $h_s \sim Exp\left(\alpha / f_s'\right), \forall s$, where

$$f_s' = \sum_{l \in A_s} f_s'^l, \ \forall s \in \mathcal{S}. \tag{27}$$

The expected value and variance of $(h_s)^n$ can then be found by:

$$E[(h_s)^n] = n \int_0^\infty t^{n-1} e^{-\frac{f_s'}{\alpha} t} dt = n! \left(\frac{\alpha}{f_s'} \right)^n, \ \forall s \in \mathcal{S}, \text{ and} \tag{28}$$

$$Var[(h_s)^n] = E[(h_s)^{2n}] - (E[(h_s)^n])^2 = \left((2n)! - (n!)^2\right) \left(\frac{\alpha}{f_s'} \right)^{2n}, \ \forall s \in \mathcal{S}. \tag{29}$$

Substituting Eqs. (28) and (29) in Eqs. (25) and (26) respectively, we obtain the expected value and variance of the congestion cost on route section s as shown below:

$$E\left[\varphi_s\right] = \beta_s n!\left(\frac{\alpha(V_s+\overline{V}_s)}{\gamma k f_s'}\right)^n,\ \forall s \in \mathcal{S},\ \text{and} \tag{30}$$

$$Var\left[\varphi_s\right] = \beta_s^{\,2}((2n)!-(n!)^2)\left(\frac{\alpha(V_s+\overline{V}_s)}{\gamma k f_s'}\right)^{2n},\ \forall s \in \mathcal{S}. \tag{31}$$

5. EFFECTIVE TRAVEL COST

The variabilities associated with the in-vehicle travel time and waiting time, coupled with the effect of congestion causes variability in route travel time. Due to this, passengers cannot determine the exact total trip time to complete their journeys. The variability in route travel time is countered by early departures to allow for additional time to avoid late arrivals. This additional time is included by the passengers while planning their trips, and is referred to as safety margin. This safety margin plus the expected trip time is known as effective travel cost or travel time budget (Lo *et al.*, 2006). Mathematically, the effective travel cost on a particular route can be formulated as

$$E_r^w = E[C_r^w] + \rho\sqrt{Var(C_r^w)}, \forall r \in \mathcal{R}_w, w \in \mathcal{W}, \tag{32}$$

where E_r^w is the effective travel cost of route *r* between OD pair *w*. C_r^w is the trip travel time (including in-vehicle travel time and waiting time) on route r connecting OD pair *w*. ρ is the parameter related to the requirement of arriving within time. For trips that have a high penalty on lateness, passengers will reserve a relatively large safety margin, or equivalently, a high value of ρ .The term $\rho\sqrt{Var(C_r^w)}$ is interpreted as the safety margin of passengers. The parameter ρ relates to the probability that the actual trip travel cost is not greater than effective travel cost:

$$P\{C_r^w \le E_r^w = E[C_r^w] + \rho\sqrt{Var(C_r^w)}\} = \lambda, \tag{33}$$

where λ is referred to as within budget time reliability or the probability that the trip travel time is not greater than the effective travel cost. Similar to Lo *et al.* (2006), it can be verified that if the Lindeberg condition is satisfied, then the route travel time C_r^w follows a normal distribution according to the Central Limit Theorem, regardless of individual independent link travel time distribution. As a result, the random variable C_r^w can be normalized as shown below:

$$P\left\{\frac{C_r^w - E[C_r^w]}{\sqrt{Var(C_r^w)}} \le \rho\right\} = \lambda. \tag{34}$$

Let $Z_{C_r^w} = \dfrac{C_r^w - E[C_r^w]}{\sqrt{Var(C_r^w)}}$ denote the standard normal variate of C_r^w and hence Eq. (34) can be written as:

$$P(Z_{C_r^w} \le \rho) = \lambda. \tag{35}$$

The parameter λ, can then be interpreted as the degree of risk aversion of passengers. A higher value of λ means that passenger is more risk-aversive, while a lower value means that passenger is more risk-prone. Thus the value of λ totally depends on the individual's appetite for risk-taking and the purpose of the trip.

The route cost, C_r^w, is related to route section costs as follows:

$$C_r^w = \sum_{s\in S} b_{sr} C_s, \ \forall r \in \mathcal{R}_w, \forall w \in \mathcal{W}. \tag{36}$$

Taking expectation and variance on both sides of Eq. (36), we get the following respectively:

$$E[C_r^w] = \sum_{s\in S} b_{sr} E[C_s], \ \forall r \in \mathcal{R}_w, \forall w \in \mathcal{W} \text{ , and} \tag{37}$$

$$Var[C_r^w] = \sum_{s\in S} b_{sr} Var[C_s], \ \forall r \in \mathcal{R}_w, \forall w \in \mathcal{W}, \tag{38}$$

where the variances of route costs are assumed to be independent of each other. The effective travel cost on route r between OD pair w can then be obtained by substituting Eqs.(37), and (38) into Eq.(32):

$$E_r^w = \sum_{s\in S} b_{sr} E[C_s] + \rho \sqrt{\sum_{s\in S} b_{sr} Var[C_s]}, \ \forall r \in \mathcal{R}_w, \forall w \in \mathcal{W}. \tag{39}$$

By taking expectation and variance on both sides of Eq. (4) respectively and substituting the resulting expressions into Eq. (39), we have:

$$\begin{aligned} E_r^w &= \sum_{s\in S} b_{sr} \left(E[T_s] + E[X_s] + E\left[\varphi_s\right]\right) \\ &\quad + \rho \sqrt{\sum_{s\in S} b_{sr} \left(Var[T_s] + Var[X_s] + Var[\varphi_s]\right)}, \ \forall r \in \mathcal{R}_w, \forall w \in \mathcal{W}, \end{aligned} \tag{40}$$

which is the travel cost function used in this paper.

6. PROBLEM FORMULATION

In a congested transit network, due to the lack of perfect information on the network condition, a variation in the passengers' perception of effective travel cost is bound to exist. This variation is modeled by a random variable called perceived effective travel cost, B_r^w, which can be expressed as a sum of effective travel cost and a random error term, ε_r^w, such that $E[\varepsilon_r^w] = 0$. Hence, $E[B_r^w] = E_r^w, \forall r \in \mathcal{R}_w, w \in \mathcal{W}$.

Assuming that all passengers choose the routes with minimum perceived effective travel cost, we define *reliability-based stochastic user equilibrium (RSUE)* condition as follows:

Definition 1: The transit network is said to be at *reliability-based stochastic user equilibrium*, if for each OD pair, the perceived effective travel cost of each route is equal and no passenger can reduce his/her perceived effective travel cost by unilaterally changing routes.

Mathematically, the RSUE condition can be expressed as:

$$y_r^w = p_r^w q_w \text{ or } y_r^w - p_r^w q_w = 0, \quad \forall r \in \mathcal{R}_w, \forall w \in \mathcal{W}, \tag{41}$$

where y_r^w is the passenger route flow on route r connecting OD pair $w \in \mathcal{W}$. q_w is the demand of OD pair $w \in \mathcal{W}$. p_r^w is the probability of selecting route $r \in \mathcal{R}_w$ between OD pair

$w \in \mathcal{W}$. The above condition ensures that the perceived effective travel cost of each route is equal.

If the random error term, ε_r^w, is assumed to be identically and independently Gumbel distributed random variables with mean zero and identical standard deviation, the route choice probability, p_r^w, for route $r \in \mathcal{R}_w, w \in \mathcal{W}$ can be given by the following logit formula:

$$p_r^w = \frac{\exp(-\theta E_r^w)}{\sum_{k' \in R_w} \exp(-\theta E_{k'}^w)}, \quad \forall r \in \mathcal{R}_w, \forall w \in \mathcal{W}, \tag{42}$$

where $\theta > 0$ is a given parameter which measures the passengers' perceptions of effective travel cost on a particular route in transit networks. A higher value of θ implies a smaller perception error.

The demand is assumed to be a decreasing function of the perceived effective travel cost S^w, which can be expressed as:

$$S^w = -\frac{1}{\theta} \log\left(\sum_{r \in R_w} \exp(-\theta E_r^w) \right), \quad \forall w \in \mathcal{W}. \tag{43}$$

In this study, the following function is adopted for the purpose of analysis:

$$q^w = q^0 \left(S^w \right)^{-e}, \quad \forall w \in \mathcal{W}, \tag{44}$$

where q^0 is the maximum potential demand and e is the demand elasticity. When $e = 0$, demand is totally inelastic.

The proposed doubly stochastic transit assignment problem can be formulated as a VI problem via the approach described in Szeto and Lo (2005):

Find $y_r^{w*} \in \Omega$ such that

$$\sum_{w \in W} \sum_{r \in R_w} \left(y_r^{w*} - p_r^{w*} q_w^{\ *} \right)\left(y_r^w - y_r^{w*} \right) \geq 0, \quad \forall y_r^w \in \Omega = R_+^{m_r}, \tag{45}$$

where $m_r = \sum_w |R_w|$ and the superscript * refers to the solution of the VI problem. p_r^w and q^w in VI (45) are defined by (1)-(3), (7), (8), (16), (19)-(26), (30), (31), (40), (42)-(44). When $\theta \to \infty$, the RSUE solution tends to the *reliability-based deterministic user equilibrium* (RDUE) solution. Further, if $\rho = 0$ and $e = 0$, then the RSUE solution tends to the de Cea and Fernández solution. Further, if $\beta_s = \beta^l = 0$, then the RSUE solution tends to the Spiess and Florian solution.

The VI problem (45) can be written in vector form as below:

$$\mathbf{E}^T(\mathbf{Y}^*)(\mathbf{Y} - \mathbf{Y}^*) \geq 0, \forall \mathbf{Y} \subseteq \Omega = R_+^{m_r}, \text{ where } \mathbf{E}(\mathbf{Y}) = \left[y_r^w - p_r^w q_w \right], \mathbf{Y} = [y_r^w]. \tag{46}$$

According to Nagurney (1999), a solution exists to the VI problem (46) when $\mathbf{E}(\mathbf{Y})$ is continuous with respect to $\mathbf{Y}$ (or E_r^w is continuous with respect to y_r^w) and the solution set is bounded and closed (i.e., compact). In addition, the solution is unique when $\mathbf{E}(\mathbf{Y})$ is strictly monotonic with respect to $\mathbf{Y}$. Clearly, the solution set is compact in this problem. It is because the route flows cannot be greater than the corresponding maximum potential demand so that the solution set must be bounded by a sphere with its radius equal to the largest demand of all OD pairs. Moreover, E_r^w is continuous with respect to y_r^w as all the functions involved for calculating E_r^w are continuous of y_r^w. Therefore, $\mathbf{E}(\mathbf{Y})$ is continuous with respect to $\mathbf{Y}$ and a solution exists to the proposed VI problem.

The VI formulation (46) can be reformulated into the following unconstrained and differentiable minimization program via the approach in Szeto *et al.* (2006):

$$\min G(\mathbf{Y}) = \frac{1}{2}\sum_{w\in W}\sum_{r\in R_w}\left[\sqrt{\left(y_r^w\right)^2 + \left(y_r^w - p_r^w q_w\right)^2} - y_r^w - \left(y_r^w - p_r^w q_w\right)\right]^2 . \tag{47}$$

When the objective function value is equal to the global minimum of zero, the optimal solution obtained is also optimal to the VI problem (46). Hence, the VI problem (46) can be solved by many existing optimization methods. In this paper, the unconstrained minimization problem (47) is solved by Premium Solver Platform.

7. NUMERICAL EXAMPLES

To illustrate the properties of the problem, numerical studies were carried out. The example network adopted was the one discussed in Section 2. The basic link data related to the network is given in Table 1. All vehicle capacities were set to 10 passengers/bus. The headway was assumed to be exponentially distributed with mean $1/f_s^l$. The above parameters were identical as the ones in de Cea and Fernández (1993), except that the variances were set arbitrarily. The other parameters were set as follows: $\beta^l = 1$, $\beta_s = 0.1$, θ = 0.1, $m = 4, n = 3$, $\alpha = \gamma = 60$ min/hr, $q^0 = 200$ pass/h, e = 0.2, and $\lambda = 95\%$, unless otherwise specified.

Table 1. Travel times and variances for route sections

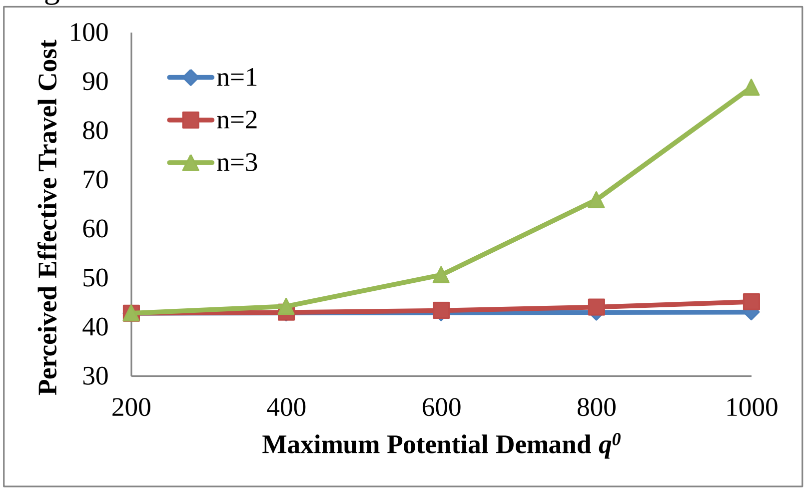
Route Section	S1	S2	S3	S4	S5	S6
Mean (min)	25	7	*5.4*	*9*	13	8
Variance (min^2)	3	12	*6.8*	*15.8*	35	14

Note: Italic fonts denote weighted average travel times or variances

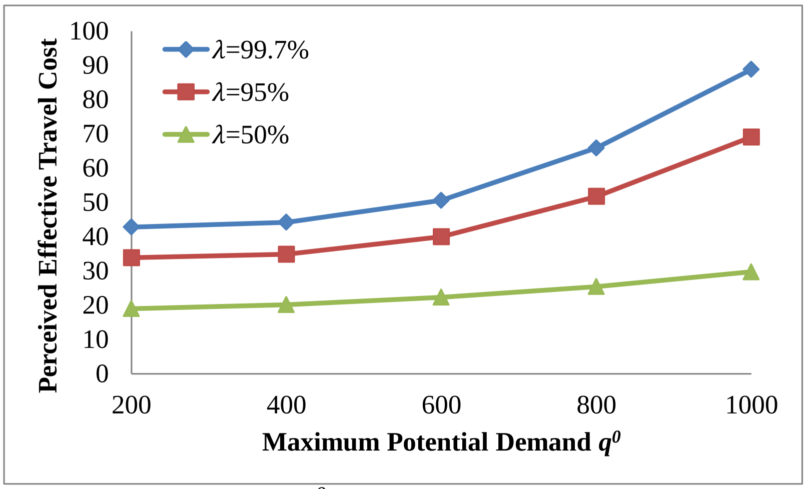
To illustrate the effects of the congestion factor parameter, *n*, and the maximum potential demand on the perceived effective travel cost, Figure 2(a) is plotted. It can be inferred that the perceived effective travel cost is monotonically increasing with the maximum potential demand for various values of *n*. This is reasonable as a higher potential demand results in higher travel demand, a more congested transit network and therefore a higher travel cost. The perceived effective travel cost is also monotonically increasing with *n* for various maximum potential demand levels. This is because for a given level of service, the value of *n* increases means that more passengers are willing to wait for the next arriving vehicle and hence the congestion cost increases. This increase results in a corresponding increase in the perceived effective travel cost and the perceived effective travel cost.

Figure 2(b) shows the perceived effective travel cost for various values of the maximum potential demand and λ. As expected, the perceived effective travel cost is monotonically increasing with the maximum potential demand for various values of λ, since the congestion cost increases with the maximum potential demand. Figure 2(c) plots the results for the perceived effective travel cost for various changes in frequency of each line and various values of λ. The x-axis is for the changes in frequency. For example, -2 represents the setting that the frequencies of all lines were reduced by 2. From this figure, we observe that the changes in the frequencies under various risk aversion behaviors have a strong influence on the perceived effective travel cost. As the frequency decreases, there is a sharp increase in the perceived effective travel cost, because a lower frequency results in higher means and variances of waiting cost and congestion cost. Moreover, the influence of frequency on the most risk-aversive passengers is the largest as they are very sensitive to the changes in mean and variance of waiting time.

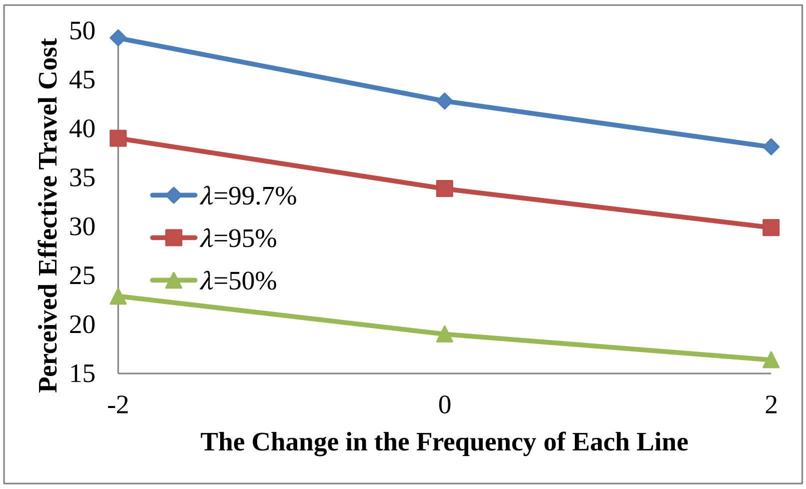
From Figures 2(b) and 2(c), it can also be observed that the perceived effective travel cost increases with the value of λ, since a more risk-aversive passenger has a larger safety margin. In particular, the passengers who fall under the category of $\lambda = 99.7\%$ reserve a larger safety margin, as can be seen in Figure 3. However, when $\lambda = 50\%$, the passengers do not reserve any safety margin because they only consider the mean travel cost and ignore the variance of travel cost. The variations in safety margin reserved by passengers with different risk-taking behaviors on all the four routes can also be observed in Figure 3. The variations are due to different travel time uncertainties on these routes. Risk aversive passengers reserve larger safety margins for more uncertain routes.



(a) Effect of various values of q^0 and n on the perceived effective travel cost



(b) Effect of various values of q^0 and λ on the perceived effective travel cost



(c) Effect of various values of frequency and λ on the perceived effective travel cost
Figure 2. Effects of various parameters on the perceived effective travel cost

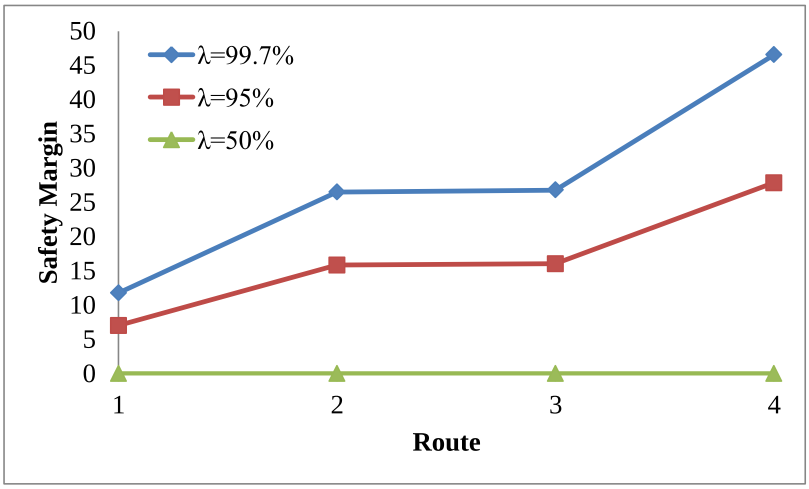


Figure 3. Safety margin for each route at different degrees of risk aversion

Table 2 presents the RSUE route flows, the mean and variance of route travel cost, and effective travel cost for passengers with different degrees of risk aversion. From these results, we can see that the degree of risk aversion has a major influence in determining the route flow pattern. For λ = 95% and 99.7%, route 1 is highly attractive when the maximum potential demand is 200 pass/h. This is because the effective travel cost is the lowest for this particular route and hence highly risk-aversive passengers choose this route. Similarly, route 2 is also attractive as the mean travel cost is the lowest and the variance is not too large, so that route 2 has a very similar effective travel cost compared with route 1. However, route 4 attracts the least flow due to its large effective travel cost. The large effective total travel cost is due to a large travel cost variance compared with the variances of other routes. However, the passenger flow on this high variance route increases, as λ decreases. The reduction in λ implies the increase in the tolerance level of passengers to arrive late at their destination and hence the increase in flow on the route with a high variance. When λ decreases to 50%, the flow on route 4 increases to the level similar to the flows on other routes, because passengers ignore the variance and the mean travel costs of all routes are similar. The implication is that ignoring the risk aversion in transit assignment can wrongly estimate the flow pattern and hence the total travel cost and the level of service of each line. Indeed, the variance of each route increases as λ decreases. This observation is consistent with the trend observed in Figure 3, because the higher is the variance, the larger is the safety margin reserved for that particular route.

Table 2. The influence of degree of risk aversion

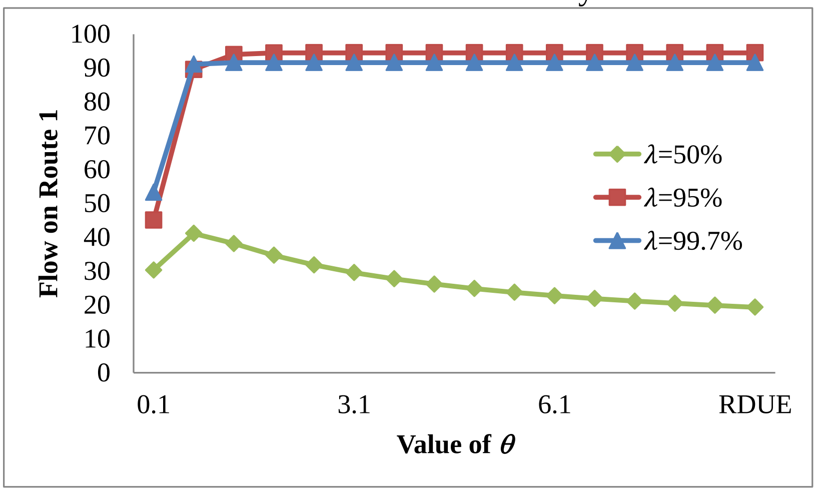
Route	Route Flow	Travel cost		E_r^w
		Mean	Variance	
λ=99.7%				
1	53.3	31.1	39.2	48.3
2	22.2	30.5	93.1	57.0
3	14.8	34.3	95.2	61.1
4	1.7	36.0	287.0	82.6
λ=95%				
1	45.1	31.1	39.1	41.3
2	27.0	30.6	93.1	46.5
3	18.2	34.3	95.3	50.4
4	4.7	36.1	287.1	64.0
λ=50%				
1	30.4	31.0	39.0	31.0
2	31.2	30.7	93.9	30.7
3	21.4	34.5	96.1	34.5
4	17.8	36.3	288.2	36.3

Table 3. Route flows and effective travel costs for various values of θ

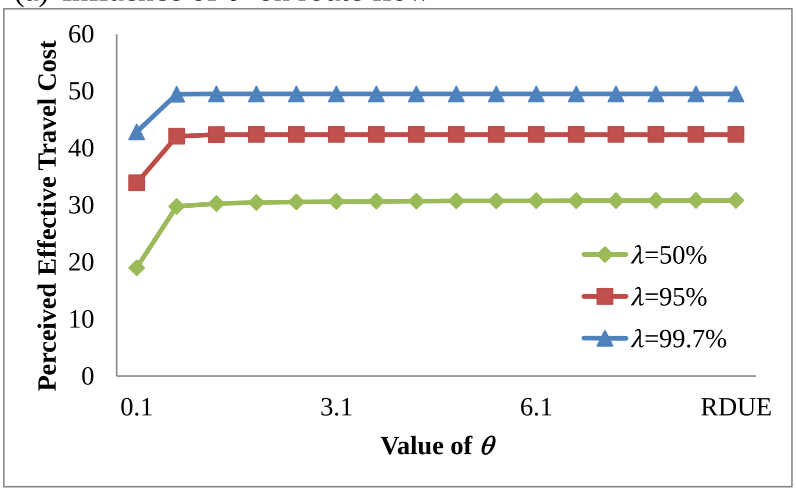
θ	0.1		1.0		RDUE	
Route	y_r^w	E_r^w	y_r^w	E_r^w	y_r^w	E_r^w
λ=99.7%						
1	53.3	48.3	91.6	49.5	91.6	49.5
2	22.2	57.0	0.0	57.0	0	57.0
3	14.8	61.1	0.0	61.0	0	61.0
4	1.7	82.6	0.0	82.6	0	82.6
λ=95%						
1	45.1	41.3	92.9	42.3	94.5	42.4
2	27.0	46.5	1.6	46.4	0	46.4
3	18.2	50.4	0.0	50.3	0	50.3
4	4.7	64.0	0.0	63.9	0	63.9
λ=50%						
1	30.4	31.0	40.0	31.0	8.6	31.0
2	31.2	30.7	59.3	30.6	92.1	31.0
3	21.4	34.5	1.4	34.4	0	34.9
4	17.8	36.3	0.2	36.1	0	36.5

Table 3 reports the route flows for various values of θ and λ. Experimental runs were carried out for $\theta = 0.1$ and 1.0 and the reliability-based deterministic user equilibrium (RDUE) cases. As θ increases, the flow on route 4 decreases, irrespective of the value of λ. A higher value of θ means better passengers' knowledge about the travel time, waiting time and their variances in the transit network and hence, the flow on route 4 with high effective travel cost is reduced. Similar inference can be drawn from Figure 4(a), which plots the flow on route 1, for various values of θ. As θ increases, the flow on route 1 also increases for λ = 95% and 99.7%, since more passengers know that route 1 is the actual best route in terms of effective travel cost. Similarly, for λ = 50%, the flows on routes 1 and 2 increases with an increasing value of θ as shown in Table 3, since more passengers know that the mean travel cost on these two routes are very close (their difference is only 0.3) and are lower than routes 3 and 4. However, when θ becomes larger, the small difference in the mean travel time of routes 3 and 4 matters, and more passengers select the least mean travel route and hence the flow on route 1 drops eventually as θ increases.

As shown in Table 3, the route flow pattern for θ = 1.0 is close to the RDUE flow pattern except that for λ = 99.7%, the two flow patterns look identical due to the truncation error. Indeed, as θ approaches to infinity, the resultant flow pattern tends to the RDUE flow pattern. One more comment is that the flows on routes 3 and 4 are exactly equal to 0 for the RDUE case but the corresponding flow is not exactly equal to 0 for the RSUE case, as the RSUE case guarantees that each route must carry some flows. Their flow values are differentiated by 0 and 0.0 respectively. On the other hand, the perceived effective travel cost is monotonically increasing with θ for various values of λ (Figure 4b) and approaches the effective travel cost of the RDUE case as θ tends to infinity.



(a) Influence of θ on route flow



(b) Influence of θ on the perceived effective travel cost

Figure 4. Influence of θ on route flow and the perceived effective travel cost

Table 4 compares five fixed demand cases (i.e., $e=0$) to illustrate the effects of ignoring congestion, the risk aversion behavior of passengers, and perceived errors on three performance measures namely, the total effective travel cost, the total mean travel cost, and the total perceived effective travel cost. Following the high congestion case in the appendix of de Cea and Fernández (1993), the parameters are set as $\beta_s = 20, m=1, n=1, q^0 = 240$. The flow solutions of the de Cea and Fernández approach and the Spiess and Florian approach obtained from that appendix were substituted into our model with λ = 99.7% to compute the performance measures in Table 4. The number in parenthesis represents the percentage change in the performance measure compared with case 1 (i.e., RSUE, θ = 0.1) and case 2 (RSUE, θ = 0.5) respectively. Case 3 is the case of RDUE. Case 4 considers the de Cea and Fernández approach that ignores the risk aversion behavior of passengers. Case 5 considers the Spiess and Florian approach that ignores both the effects of congestion and the risk aversion behavior of passengers on transit assignment. It can be learnt from the table that case 3 underestimates the total effective travel cost, while it overestimates the total perceived travel cost. Compared with case 1, case 3 underestimates the total perceived travel cost greatly (more than 6%). Cases 4 and 5 overestimate all three performance measures as compared to the cases 1 and 2. In the extreme case, their approaches overestimate the total perceived effective travel cost by more than 14%. Because of ignoring the effects of congestion and risk aversion behavior, case 3 and 4 models overestimate the flows on some routes with high congestion levels or variances, which is unrealistic and inaccurate.

Table 4. A comparison of performance measures calculated via different approaches

	Case 1: RSUE, θ = 0.1	Case 2: RSUE, θ = 0.5	Case 3: RDUE	Case 4: The de Cea and Fernández approach	Case 5: The Spiess and Florian approach
Total mean travel cost	14384.2	14181.9	14182.7 (-1.4%, 0.01%)	14487.6 (0.7%, 2.2%)	15006.7 (4.3%, 5.8%)
Total effective travel cost	31923.9	31554.9	31521.8 (-1.3%, -0.1%)	32859.1 (2.9%, 4.1%)	33818.4 (5.9%, 7.2%)
Total perceived effective travel cost	29617.2	31177.2	31521.8 (6.4%, 1.1%)	32859.1 (10.9%, 5.4%)	33818.4 (14.2%, 8.5%)

8. CONCLUSIONS

In this paper, a VI formulation is proposed for reliability-based stochastic transit assignment problems with elastic demand. Compared with Spiess and Florian's and de Cea and Fernández's models, the VI formulation captures the effect of congestion, perceived errors on travel times, and the risk-aversive behavior of passengers, and includes their models as special cases. In-vehicle travel time, waiting time, and the additional waiting due to congestion are considered as stochastic variables and their means and variances are used to define the reliability-based stochastic user equilibrium condition. To illustrate the properties of the proposed model, sensitivity studies were carried out. A numerical study was also set up to show that Spiess and Florian's and de Cea and Fernández's models can overestimate the system performance measures. This seems to indicate that when most passengers in congested transit networks are risk-averse and have imperfect information on transit performance, transit assignment models that can capture these two issues including ours should be used to predict the patronage of transit lines to have more accurate and realistic results.

The proposed formulation considers a constant capacity for all the transit vehicles in transit network. While this is not realistic, it is not difficult to extend the formulation so that it can accommodate vehicles of different types and different capacities. We leave this to future studies. The formulation proposed here incorporates only single-class passengers. As a topic of further interest, multiple user classes can be considered under the given setting. Moreover, the assumption of exponential headway distribution is realistic to the transit stops without dynamic passenger information systems but may not be realistic to the stops with these systems. In the future, one can extend the proposed framework in this paper to consider the realistic assumption mentioned in Nökel and Wekeck (2009) for transit assignment under the provision of dynamic passenger information systems. Furthermore, this formulation does not consider demand uncertainties. Extending this formulation to consider both demand and supply uncertainties is one important future research direction. In addition, the proposed problem formulation was solved using commercially available general-purpose software. The special structure of the problem has not been used to solve for solutions efficiently. One may investigate the special structure to develop an efficient and convergent solution method in the future. In addition, based upon the model proposed here, we can develop transit network design models with uncertainty consideration.

ACKNOWLEDGEMENT

The work described in this paper was partially supported by a grant from the Central Policy Unit of the Government of the Hong Kong Special Administrative Region and the Research Grants Council of the Hong Kong Special Administrative Region, China (HKU7026-PPR-12), a grant (201011159026) from the University Research Committee, a grant from the National Natural Science Foundation of China (71271183), and a Research Postgraduate Studentship from the University of Hong Kong. The authors are grateful to the two reviewers for their constructive comments.

REFERENCES

Abdel-Aty, M.A., Kitamura, R., Jovanis, P.P. (1997) Using stated preference data for studying the effect of advanced traffic information on drivers' route choice. *Transportation Research Part C*, 5(1), 39-50.

Cepeda, M., Corninetti, R., Florian, M. (2006) A frequency-based assignment model for congested transit networks with strict capacity constraints: characterization and computation of equilibria. *Transportation Research Part B*, 40(6), 437-459.

Chen, B.Y., Lam, W.H.K., Sumalee, A., Shao, H. (2011) An efficient solution algorithm for solving multi-class reliability-based traffic assignment problem. *Mathematical and Computer Modelling*, 54, 1428-1439.

Cominetti, R., Correa, J. (2001) Common-lines and passenger assignment in congested transit networks. *Transportation Science*, 35(3), 250-267.

de Cea, J., Fernández, E. (1993) Transit assignment for congested public transport systems: an equilibrium model. *Transportation Science*, 27(2), 133-147.

Fu, X., Lam, W.H.K., Chen, B.Y. (2012) A reliability-based traffic assignment model for multi-modal transport network under demand uncertainty. *Journal of Advanced Transportation*, DOI:10.1002/atr.202.

Hamdouch, Y., Ho, H.W., Sumalee, A., Wang, G. (2011) Schedule-based transit assignment model with vehicle capacity and seat availability. *Transportation Research Part B*, 45, 1805-1830.

Hamdouch, Y., Lawphongpanich, S. (2008) Schedule-based transit assignment model with travel strategies and capacity constraints. *Transportation Research Part B*, 42(7-8), 663-684.

Jackson, W.B., Jucker, J.V. (1982) An empirical study of travel time variability and travel choice behavior. *Transportation Science*, 16(4), 460-475

Lam, W.H.K., Gao, Z.Y., Chan, K.S., Yang, H. (1999) A stochastic user equilibrium assignment model for congested transit networks. *Transportation Research Part B*, 33(5), 351-368.

Lam, W.H.K., Zhou, J., Sheng, Z.H. (2002) A capacity restraint transit assignment with elastic line frequency. *Transportation Research Part B*, 36(10), 919-938.

Li, Z.C., Lam, W.H.K., Sumalee, A. (2008) Modeling impacts of transit operator fleet size under various market regimes with uncertainty in network. *Transportation Research Record, Journal of Transportation Research Board*, 2063, 18-27.

Li, Z.C., Lam, W.H.K., Wong, S.C. (2009) The optimal transit fare structure under different market regimes with uncertainty in the network. *Networks and Spatial Economics*, 9, 191-216.

Lo, H.K., Luo, X.W., Siu, B.W.Y. (2006) Degradable transport network: travel time budget of travelers with heterogeneous risk aversion. *Transportation Research Part B*, 40(9), 792-806.

Nagurney, A. (1999) *Network Economics: A Variational Inequality Approach.* Kluwer Academic Publishers, USA.

Nguyen, S., Pallottino, S., Malucelli, F. (2001) A modeling framework for passenger assignment on a transport network with timetables. *Transportation Science*, 35(3), 238-249.

Nökel, K., Wekeck, S. (2009) Boarding and alighting in frequency-based transit assignment. *Transportation Research Record, Journal of Transportation Research Board*, 2111, 60-67.

Nuzzolo, A., Russo, F., Crisallir, U. (2001) A doubly dynamic schedule-based assignment model for transit networks. *Transportation Science*, 35(3), 268-285.

Poon, M.H., Wong, S.C., Tong, C.O. (2004) A dynamic schedule-based model for congested transit networks. *Transportation Research Part B*, 38(4), 343-368.

Shao, H., Lam, W.H.K., Tam, M. (2006) A reliability-based stochastic traffic assignment model for network with multiple user classes under uncertainty in demand. *Networks and Spatial and Economics*, 6(3), 173-204.

Spiess, H., Florian, M. (1989) Optimal Strategies: a new assignment model for transit networks. *Transportation Research Part B*, 23(2), 83-102.

Sumalee, A., Tan, Z.J., Lam, W.H.K. (2009) Dynamic stochastic transit assignment with explicit seat allocation model. *Transportation Research Part B*, 43(8-9), 895-912.

Sumalee, A., Uchida, K. and Lam, W.H.K. (2011) Stochastic multi-modal transport network under demand uncertainties and adverse weather condition. *Transportation Research Part C*, 19, 338-350.

Szeto, W.Y., Jiang, Y., Wong, K.I., Solayappan, M. (2011b) Reliability-based stochastic transit assignment with capacity constraints: Formulation and solution method. *Transportation Research Part C,* in press.

Szeto, W.Y., Lo, H.K. (2005) The impact of advanced traveler information services on travel time and schedule delay costs. *Journal of Intelligent Transportation Systems: Technology, Planning, and Operations*, 9(1), 47-55.

Szeto, W.Y., O'Brien, L., O'Mahony, M. (2006) Risk-averse traffic assignment with elastic demands: NCP formulation and solution method for assessing performance reliability. *Network and Spatial Economics*, 6(3-4), 313-332.

Szeto, W.Y., Solayappan, M., Jiang, Y. (2011a) Reliability-based transit assignment for congested stochastic transit networks. *Computer Aided Civil and Infrastructure Engineering*, 26(4), 311-326.

Teklu, F. (2008) A stochastic process approach for frequency-based transit assignment with strict capacity constraints. *Networks and Spatial Economics*, 8(2), 225-240.

Tong, C.O., Wong, S.C. (1998) A stochastic transit assignment model using a dynamic schedule-based network. *Transportation Research Part B*, 33(2), 107-121.

Yang, L., Lam, W.H.K. (2006) Probit-type reliability-based transit network assignment. *Transportation Research Record, Journal of Transportation Research Board*, 1977, 154-163.