

A Column Generation-based Algorithm for Multi-Class Dynamic User Equilibrium Problem

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Abstract: The multi-class dynamic user equilibrium (MDUE) model can be adopted in the lower level of a bi-level formulation of dynamic network design problems to predict heterogeneous travelers' path choices in response to transportation authority's decisions in the upper level. While a number of previous studies have devoted to the finite-dimensional MDUE (FMDUE) problem with a given number of user classes, the infinite-dimensional MDUE (IMDUE) problem was rarely addressed. This work proposes a column generation-based algorithm that solves the IMDUE problem as a series of FMDUE sub-problems. Each sub-problem is formulated as a nonlinear minimization program via a gap function and solved by a feasible descent direction method that is able to circumvent the needs to calculate partial derivatives for determining search direction, while maintaining the mechanisms of searching along feasible descent directions. Numerical results show our approach is superior to the algorithm based on the method of successive averages.

Keywords: Traffic assignment; dynamic user equilibrium; column generation; user heterogeneity.

1. INTRODUCTION

Dynamic user equilibrium (DUE, Smith, 1993) models have been widely applied to obtain time-varying link or path flows in response to supply- or demand-side traffic network improvement and management measures, such as adding new roads (or lanes), ramp metering, managed lanes, congestion pricing, signal optimization, and information provision (FHWA, 2012). Particularly, DUE models can be adopted in the lower level of bi-level formulations of dynamic network design problems (e.g., Clegg et al., 2001; Yang et al., 2003; Smith, 2005; Joksimovic et al., 2005; Karoonsoontawong and Waller, 2007; Chen and Hu, 2010) to predict users' responses in path (and/or departure time) choices to transportation authority's decisions in the upper level. As opposed to most existing DUE models that considered homogeneous users in a road network, a few of previous studies explicitly took heterogeneous users' characteristics and preferences (e.g., income, value of time, risk-taking behavior, information availability, etc.) into account and developed multi-class or multi-criterion DUE models (e.g., Peeta and Mahmassani, 1995; Huang and Lam, 2003; He et al., 2003; Yang et al., 2003; Joksimovic et al., 2006; Lu et al., 2008), in order to realistically describe travelers' choice behavior.

This work deals with the infinite-dimensional multi-class DUE (IMDUE) problem which considers each traveler poses a distinct characteristic or preference of interest in the underlying path choice decision framework. While a number of previous studies have devoted to the

finite-dimensional MDUE (FMDUE) problem with a predetermined number of user classes (e.g., Peeta and Mahmassani, 1995; Huang and Lam, 2003; He et al., 2003; Yang et al., 2003; Joksimovic et al., 2006), the IMDUE problem was rarely addressed in the literature. Recognizing that the evaluation of congestion pricing strategies requires modeling the response of users with non-identical preferences, Lu et al. (2008) developed a IMDUE model, called bi-criterion DUE or BDUE, which explicitly considers that tripmakers with different values of time (VOT) select paths that simultaneously optimize the two essential choice criteria: travel time and out-of-pocket cost (e.g., Dial, 1979). As opposed to conventional DUE models that designated a constant VOT or a discrete set of VOT to all tripmakers in a network (e.g., Uchida and Sugiki, 2011), the BDUE model allows the VOT to be continuously distributed among tripmakers. Numerical results demonstrated that the VOT distribution significantly affects path flow patterns and toll road usage, highlighting the necessity of addressing user heterogeneity in dynamic traffic assignment (DTA) models for time-varying road pricing.

Lu et al. (2008) presented an algorithm based on the method of successive averages (MSA) to solve the IMDUE problem. The MSA remains by far one of the most widely used solution heuristics in DTA context, due to its simplicity and non-requirement of derivative information for the flow-cost mapping function (e.g., Oh and Park, 2011). However, the MSA does not guarantee descent (or improvement in the objective function) at every iteration (Bertsekas, 1995). Moreover, its convergence properties in real-life networks have been inconclusive, mainly because pre-determined and across-the-board step sizes do not exploit local information in searching for a solution, and therefore tend to have sluggish performance properties (Lu et al., 2009).

To provide a more effective solution approach than MSA-based approaches for the IMDUE problem, this work formulates the problem as infinite-dimensional variational inequalities and proposes an algorithm based on the idea of column generation method, or Dantzig-Wolf decomposition (e.g., Dantzig and Wolfe, 1960; Larsson and Patriksson, 1992; Patriksson, 1994), which solves the IMDUE problem as a series of FMDUE sub-problems and progressively finds (approximate) solutions. Specifically, integrated in the column generation-based algorithmic framework are (i) a parametric-based path generation scheme that partitions the entire range of VOT into many subintervals and accordingly determines the multiple user classes and the corresponding least generalized cost (or extreme non-dominated) paths for each user class (Mahmassani et al., 2005), (ii) a feasible descent direction method for updating multi-class path flows, and (iii) a multi-class dynamic network loading (MDNL) model that captures traffic dynamics and determines experienced path travel times for a given path flow pattern. Numerical experiments will be conducted on two large-scale road networks to compare the performance of our approach and that of the MSA-based heuristic proposed by Lu et al. (2008).

The rest of this paper is organized as follows. Section 2 presents the problem statement and formulation of the IMDUE problem, followed by the overview of the column generation-based algorithm in section 3. Section 4 describes the reduced FMDUE sub-problem and the multi-class path flow updating scheme. Section 5 reports the results of the numerical experiments which compare the solution quality of the new algorithm and the MSA in solving the problem. Concluding remarks are in section 6.

2. PROBLEM STATEMENT AND MODEL FORMULATION

2.1 Problem Statement

Consider a network $G = (N, A)$, where N is the set of nodes and A is the set of directed links $a = (i, j)$, $i \in N$ and $j \in N$. The time period of interest (planning horizon) is discretized into a set of small time intervals, $S = \{t_0, t_0 + \sigma, t_0 + 2\sigma, \dots, t_0 + M\sigma\}$, where t_0 is the earliest possible departure time from any origin node, σ a small time interval, and M a large number such that the intervals from t_0 to $t_0 + M\sigma$ cover S . d'_a and c'_a denotes the travel time and out-of-pocket cost for a vehicle to traverse link a in time interval t . The following notation and variables are used in this paper.

o	subscript for an origin node, $o \in N$.
d	subscript for a destination node, $d \in N$.
τ	superscript for a departure time interval, $\tau \in S$.
α	value of time (VOT); $\alpha \in [\alpha^{\min}, \alpha^{\max}]$, where α^{\min} and α^{\max} are smallest and largest possible values of α , respectively.
$P(o, d, \tau)$	the set of feasible paths for a given triplet (o, d, τ) .
p	subscript for a path $p \in P(o, d, \tau)$.
$h_{od}^\tau(\alpha)$	number of trips with VOT α departing from o to d in time interval τ .
$r_{odp}^\tau(\alpha)$	number of trips with VOT α departing from o to d in time interval τ that are assigned to path $p \in P(o, d, \tau)$.
$r(\alpha)$	the class-specific time-varying path flow vector for the trips with VOT α ; i.e. $r(\alpha) \equiv \{r_{odp}^\tau(\alpha), \forall o, d, \tau, \text{ and } p \in P(o, d, \tau)\}$.
r	the multi-class, time-varying path flow vector for the trips with all possible values of time; i.e. $r \equiv \{r(\alpha), \forall \alpha \in [\alpha^{\min}, \alpha^{\max}]\}$.
TT_{odp}^τ	experienced path travel time for the trips departing from o to d in time interval τ assigned to path $p \in P(o, d, \tau)$.
TT	vector of experienced path times; $TT = \{TT_{odp}^\tau, \forall o, d, \tau, \text{ and } p \in P(o, d, \tau)\}$.
TC_{odp}^τ	experienced path travel cost for the trips departing from o to d in time interval τ assigned to path $p \in P(o, d, \tau)$.
TC	vector of experienced path costs; $TC = \{TC_{odp}^\tau, \forall o, d, \tau, \text{ and } p \in P(o, d, \tau)\}$.

The experienced path generalized cost evaluated at a multi-class, time-varying path flow vector r and perceived by the tripmakers (or trips) with VOT α departing from o to d in time interval τ assigned to path $p \in P(o, d, \tau)$ is defined as:

$$GC_{odp}^\tau(\alpha, r) = TC_{odp}^\tau + \alpha \times TT_{odp}^\tau, \quad (1)$$

where $TT_{odp}^\tau = \sum_{(i,j,t) \in p} d'_{ij}$ and $TC_{odp}^\tau = \sum_{(i,j,t) \in p} c'_{ij}$. The VOT relative to each trip represents how much money the tripmaker is willing to trade for a unit time saving. To realistically reflect heterogeneity of the population, this study assumes VOT is continuously distributed across the population of trip-makers, with a given density function: $\phi(\alpha) > 0$, $\forall \alpha \in [\alpha^{\min}, \alpha^{\max}]$ and

$\int_{\alpha^{\min}}^{\alpha^{\max}} \varphi(\alpha) d\alpha = 1$, where the feasible range of VOT is given by the closed interval $[\alpha^{\min}, \alpha^{\max}]$. Note that the distribution of VOT is assumed known, and can be estimated from survey data (e.g. Small et al., 2005). The time-dependent origin-destination (O-D) demands for the entire feasible range of VOT over the planning horizon (i.e. $h_{od}^{\tau}(\alpha), \forall o, d, \tau$, and $\alpha \in [\alpha^{\min}, \alpha^{\max}]$) are also assumed known, a priori.

The key behavioral assumption made for the path choice decision is that each tripmaker will choose a path that minimizes the path generalized cost function, defined in Eq.(1). Specifically, for trips with VOT α , a path $p^* \in P(o, d, \tau)$ will be selected if and only if $GC_{odp^*}^{\tau}(\alpha, r) = \min_{p \in P(o, d, \tau)} GC_{odp}^{\tau}(\alpha, r)$. Based on this assumption, the IMDUE is defined as follows:

For each OD pair and for each departure time interval, every trip-maker cannot decrease the experienced path generalized cost with respect to that trip-maker's particular VOT by unilaterally changing path.

This implies that each trip-maker is assigned to a path with the least generalized cost with respect to his/her own VOT.

Based on the above definition, the IMDUE conditions can be mathematically stated as the following: $\forall \alpha \in [\alpha^{\min}, \alpha^{\max}]$,

$$r_{odp}^{\tau *}(\alpha)[GC_{odp}^{\tau}(\alpha, r^*) - \pi_{od}^{\tau}(\alpha, r^*)] = 0, \forall o, d, \tau, p \in P(o, d, \tau), \quad (2)$$

$$GC_{odp}^{\tau}(\alpha, r^*) - \pi_{od}^{\tau}(\alpha, r^*) \geq 0, \forall o, d, \tau, p \in P(o, d, \tau), \quad (3)$$

$$\sum_{p \in P(o, d, \tau)} r_{odp}^{\tau}(\alpha) = h_{od}^{\tau}(\alpha), \forall o, d, \tau, \quad (4)$$

$$r_{odp}^{\tau}(\alpha) \geq 0, \forall o, d, \tau, p \in P(o, d, \tau), \quad (5)$$

where $r^* = \{r_{odp}^{\tau *}(\alpha)\}$ is a multi-class time-varying MDUE path flow vector, and $\pi_{od}^{\tau}(\alpha, r^*)$ is the time-varying minimum O-D generalized travel cost, evaluated at r^* , for the trips with the same (o, d, τ, α) . Given the above assumptions and definition, the IMDUE problem aims to obtain a time-varying path flow vector $r_{odp}^{\tau}(\alpha), \forall o, d, \tau, p \in P(o, d, \tau)$ and $\forall \alpha \in [\alpha^{\min}, \alpha^{\max}]$, satisfying the above IMDUE conditions, for a given set of time-varying link tolls.

2.2 Model Formulation

Let $\Omega(\alpha) \equiv \{r(\alpha)\}$ be the set of feasible class-specific path flow vectors $r(\alpha)$ satisfying the path flow conservation constraints (4) and non-negativity constraints (5). The following proposition gives the equivalent VI formulation of the IMDUE problem of interest.

Proposition 1: Solving for the IMDUE flow pattern r^* is equivalent to finding the solution of a system of variational inequalities; that is, $\forall \alpha \in [\alpha^{\min}, \alpha^{\max}]$, find $r^*(\alpha) \in \Omega(\alpha)$ such that

$$\sum_{o \in O} \sum_{d \in D} \sum_{\tau=1}^T \sum_{p \in P(o, d, \tau)} GC_{odp}^{\tau}(\alpha, r^*) \times (r_{odp}^{\tau}(\alpha) - r_{odp}^{\tau}(\alpha)^*) \geq 0, \forall r(\alpha) \in \Omega(\alpha), \quad (6)$$

or in the following vector form for simplicity and clarity:

$$GC(\alpha, r^*)^T \circ (r(\alpha) - r^*(\alpha)) \geq 0, \forall r(\alpha) \in \Omega(\alpha), \text{ and } \forall \alpha \in [\alpha^{\min}, \alpha^{\max}], \quad (7)$$

where $GC(\alpha, r^*)$ is the path generalized cost vector perceived by the trips with VOT α and evaluated at flow pattern r^* , and \circ denotes the inner product of the two vectors: $GC(\alpha, r^*)$ and $(r^*(\alpha) - r(\alpha))$. Since (6) or (7) is only required to hold on $[\alpha^{\min}, \alpha^{\max}]$, it can be further represented by the following infinite-dimensional VI (see e.g. Marcotte and Zhu, 1997): find $r^* \equiv \{r^*(\alpha), \forall \alpha \in [\alpha^{\min}, \alpha^{\max}]\}$ and $r^* \in \Omega$ such that

$$GC(r^*)^T \circ (r - r^*) \geq 0, \forall r \in \Omega \quad (8)$$

where $GC(r^*) \equiv \{GC(\alpha, r^*), \forall \alpha \in [\alpha^{\min}, \alpha^{\max}]\}$, and $\Omega = \{r\} = \{\Omega(\alpha), \forall \alpha \in [\alpha^{\min}, \alpha^{\max}]\}$.

Proof: The proof of Proposition 1 is given in the Appendix. (e.g., Ran and Boyce, 1996)

Although the theoretical guarantee of properties such as existence and uniqueness of solutions to the infinite-dimensional VI problem (8) can be analytically derived, it typically requires the generalized path cost function, i.e. $GC(r)$, to be continuous and strictly monotone with respect to path flows on the finite and convex compact set Ω (e.g. Smith 1993; Nagurney 1998). Those properties of path cost functions might not be satisfied in general road networks with complex traffic controls. Moreover, the focus of this research is on developing simulation-based heuristic that solves the original IMDUE problem as a series of finite-dimensional nonlinear minimization programs (i.e., FMDUE sub-problems) to progressively find approximate solutions in large-scale road networks. Thus, further discussion of solution existence and uniqueness is beyond the scope of the current paper.

3. SOLUTION ALGORITHM

3.1 Overview of the Column Generation-Based Algorithmic Framework

To avoid explicit enumeration of all possible paths, this work develops a column generation-based heuristic that generates a representative subset of paths with competitive generalized costs and augments the path set as needed. The proposed algorithm operates as follows (Fig. 1).

In each (outer loop) iteration k , the path generation algorithm, parametric analysis method (PAM, Mahmassani et al., 2005) is applied to augments the current subset of feasible paths; newly generated paths, if any, are added to the subset. The PAM partitions the entire range of VOT into many subintervals and accordingly determines the corresponding multiple user classes and the least generalized cost (i.e., extreme non-dominated) path tree for each user class. The algorithm terminates if no new path is found for any user class, or a preset convergence criterion is satisfied; otherwise, it uses a multi-class path-swapping descent direction method to solve the (reduced) FMDUE problem defined by the current set of paths, before returning to the path generation step. This method proceeds iteratively and forms the inner loop (with iteration counter l) in the column generation-based framework. Also embedded in this algorithmic framework is

the MDNL model based on the mesoscopic traffic flow simulator developed by Jayakrishnan et al. (1994a) to determine experienced travel times for a given time-varying path flow pattern r ; that is, traffic flow propagation and the vehicular spatial and temporal interactions are addressed through the traffic simulation. By and large, the original IMDUE problem is solved in this algorithmic framework as a series of FMDUE sub-problems to progressively find approximate solutions.

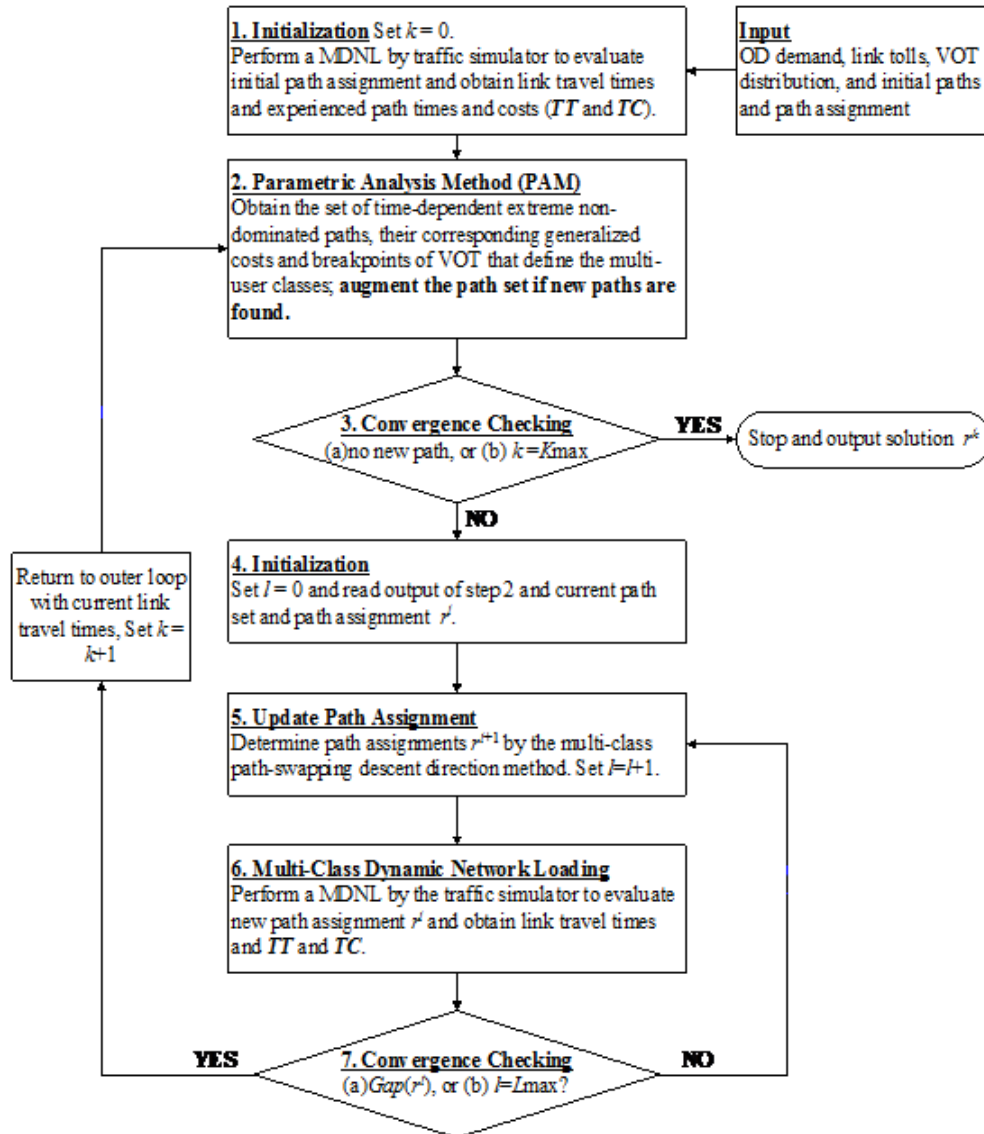


Fig. 1 Flow chart of the solution algorithm

3.2 The Parametric Analysis Method (PAM)

In order to circumvent the difficulty of finding and storing the least generalized cost path for each individual tripmaker with different VOT, the PAM is adopted to find the set of extreme non-dominated path trees, each of which minimizes the parametric path generalized cost function Eq.(1) for a particular VOT subinterval. The idea of finding the set of extreme non-dominated paths is based on the assumption that in the disutility minimization-based path choice modeling

framework with convex disutility functions, all trips would choose only among the set of extreme non-dominated paths corresponding to the extreme points on the efficient frontier in the criterion space (see e.g. Dial, 1996, 1997; Marcotte and Zhu, 1997).

Based on the parametric analysis, the algorithm is able to not only sequentially enumerate all possible time-dependent extreme non-dominated path trees (and all corresponding sensitivity ranges of VOT) but also directly move from one extreme non-dominated path tree (and its sensitivity range of VOT) to the next one without redundant calculations on the non-extreme solutions. In each iteration, the PAM is applied to obtain the set of VOT breakpoints: $\alpha = \{\alpha^0, \alpha^1, \dots, \alpha^B \mid \alpha^{\min} = \alpha^0 < \alpha^1 < \dots < \alpha^b < \dots < \alpha^B = \alpha^{\max}\}$ that partitions the entire feasible range of VOT into B subintervals: $[\alpha^{b-1}, \alpha^b)$, $b = 1, \dots, B$, and hence defines the B master user classes of trips, each master user class $u(b)$ of which covers the trips with VOT $\alpha \in [\alpha^{b-1}, \alpha^b)$. Associated with each VOT subinterval b (or master user class $u(b)$) is the time-dependent extreme non-dominated path tree: $Tr(b)$, which optimizes the path generalized cost function Eq.(1) for the corresponding VOT subinterval $[\alpha^{b-1}, \alpha^b)$ and consists of time-dependent least generalized cost paths from a given origin node, for all departure time intervals, to all the other (destination) nodes in a network.

In the column generation-based solution framework, if there is not any new path found for each (o, d, τ) and each user class $u(b)$, or the outer loop iteration counter k equals K_{max} (maximum number of outer iterations) then the algorithm terminate; otherwise it starts the inner loop with the set of VOT breakpoints (α) , as well as current path set and path assignment r^k .

4. SOLVING THE FMDUE PROBLEM

4.1 The FMDUE Problem

With the set of VOT breakpoints (α) determined by the PAM in an (outer loop) iteration k of the column generation-based algorithmic framework, the entire population of heterogeneous tripmakers in a network can be divided into a finite number of user classes, and hence the IMDUE problem can be reduced to the FMDUE problem, in which the equilibration within each user class is sought. Note that the FMDUE in each iteration k is determined based on the *current subset* of feasible paths, while the original IMDUE is defined by the complete set of feasible paths: $P(o, d, \tau)$, $\forall o, d, \tau$. In the inner loop (corresponding to an outer loop iteration k) of the column generation-based algorithmic framework, we solve the corresponding FMDUE problem which aims at finding a multi-class path flow vector that satisfies the FMDUE condition:

For each user class, each OD pair, and each departure time interval, every trip cannot decrease the experienced path generalized cost by unilaterally changing paths.

The FMDUE condition implies that, tripmakers in each user class are assigned to their respective least generalized cost path; more costly paths are not used.

The following notation and variables are defined (or redefined) for the FMDUE problem.

- (b, o, d, τ) combination of user class $u(b)$, O-D pair (o, d) and departure interval τ .
- $P(b, o, d, \tau)$ current subset of feasible time-dependent extreme non-dominated paths for a (b, o, d, τ) .
- $h_{od}^\tau(b)$ number of class $u(b)$ trips departing from o to d in time interval τ .

$r_{odp}^\tau(b)$	number of class $u(b)$ trips departing from o to d in time interval τ and assigned to path $p \in P(b, o, d, \tau)$.
$r(b)$	$\equiv \{r_{odp}^\tau(b), \forall o, d, \tau, p \in P(b, o, d, \tau)\}$; the class-specific path flow vector for the class $u(b)$ trips.
r	$\equiv \{r(b), b = 1, \dots, B\}$; the multi-class path flow vector.
$GC_{odp}^\tau(b, r)$	the path generalized cost of class $u(b)$ trips departing from o to d in time interval τ that are assigned to path $p \in P(b, o, d, \tau)$.

The FMDUE condition can then be mathematically stated as the following: for each user class b ,

$$r_{odp}^{\tau*}(b)[GC_{odp}^\tau(b, r^*) - GC_{odp^*}^\tau(b, r^*)] = 0, \quad \forall o, d, \tau, p \in P(b, o, d, \tau), \quad (9)$$

$$GC_{odp}^\tau(b, r^*) - GC_{odp^*}^\tau(b, r^*) \geq 0, \quad \forall o, d, \tau, p \in P(b, o, d, \tau), \quad (10)$$

$$\sum_{p \in P(b, o, d, \tau)} r_{odp}^{\tau*}(b) = h_{od}^\tau(b), \quad \forall o, d, \tau, \quad (11)$$

$$r_{odp}^{\tau*}(b) \geq 0, \quad \forall o, d, \tau, p \in P(b, o, d, \tau), \quad (12)$$

where r^* is a multi-class time-varying FMDUE path flow vector, and p^* denotes the (referenced) least generalized cost path of a (b, o, d, τ) : $GC_{odp^*}^\tau(b, r) = \min\{GC_{odp}^\tau(b, r), \forall p \in P(b, o, d, \tau)\}$.

To measure the deviation of a path flow vector r from the above FMDUE condition, this study extends the gap function, defined by Lu et al. (2009), to the multi-class context as the following.

$$Gap(r) = \sum_b \sum_o \sum_d \sum_\tau \sum_{p \in P(b, o, d, \tau)} r_{odp}^\tau(b) \times [GC_{odp}^\tau(b, r) - GC_{odp^*}^\tau(b, r)] \quad (13)$$

$Gap(r)$ provides a measure of the violation of the FMDUE condition in terms of the difference between the total experienced path cost and the total shortest path cost evaluated at a given path flow pattern r . The difference vanishes when the time-varying path flow vector r^* satisfies the FMDUE condition. Thus, solving the FMDUE problem can be viewed as a process of finding the path flow vector r^* such that $Gap(r^*) = 0$. Accordingly, this study formulates the FMDUE problem as a nonlinear minimization problem (NMP) by using the gap function.

$$(NMP) \quad \underset{r}{Min} \quad Gap(r) = \sum_b \sum_o \sum_d \sum_\tau \sum_{p \in P(b, o, d, \tau)} r_{odp}^\tau(b) \times [GC_{odp}^\tau(b, r) - GC_{odp^*}^\tau(b, r)] \quad (14a)$$

$$\text{Subject to } GC_{odp}^\tau(b, r) - GC_{odp^*}^\tau(b) \geq 0, \quad \forall b, o, d, \tau, p \in P(b, o, d, \tau), \quad (14b)$$

$$\sum_{p \in P(b, o, d, \tau)} r_{odp}^\tau(b) = h_{od}^\tau(b), \quad \forall b, o, d, \tau, \quad (14c)$$

$$r_{odp}^\tau(b) \geq 0, \quad \forall b, o, d, \tau, p \in P(b, o, d, \tau). \quad (14d)$$

4.2 Multi-class path flow updating/equilibrating scheme

Several conventional gradient-based solution algorithms for constrained nonlinear programming (NLP) problems could be applied to solve the NMP (Bertsekas, 1995). For a comprehensive review, the reader is referred to the book by Patriksson (1994). With the promising computational

results reported in the literature (e.g., Florian and Nguyen, 1974; Bertsekas and Gafni, 1983; Jayakrishnan et al., 1994b; Patriksson, 1994), it seems valuable to extend the gradient-based algorithms to solve the NMP. However, several issues have hindered the direct extension. For instance, evaluating first-order partial derivatives (i.e., the gradient) for determining search directions and performing line search (e.g., bisection or golden section) for obtaining optimal step sizes are computationally intensive (or intractable) for large network applications, because of the temporal dimension. Furthermore, when experienced path costs are obtained through the (simulation-based) dynamic network loading model, analytical calculations of partial derivatives are not available, and stability and accuracy of numerically calculated derivatives are not guaranteed.

Another major challenge is the presence of the constraints in Eq.14(c), given that both path flows r and least generalized path costs $GC_{odp}^r(b, r)$, $\forall (b, o, d, \tau, p)$ are the decision variables. That is, to maintain the feasibility of the updated path flows, one has to explicitly keep track of the exact change of the least generalized path costs GC^* and the set of active constraints. Standard nonlinear programming theory (e.g. Bertsekas, 1995) would suggest that the use of the Armijo step size rule in a line search scheme can help to identify the active constraints in a finite number of iterations. Besides, one can also use the gradient information to estimate the possible changes in the least travel times and the active constraint set. For example, to solve a reformulation of the static fixed demand traffic assignment problem, Lo and Chen (2000) determined search directions by using the gradient of a gap function in which partial derivatives were taken with respect to both path flows and least travel times. Nevertheless, in the simulation-based DTA model, performing a line search scheme and calculating the gradient of the gap function are computationally intensive (and prohibitive in real networks). To deal with this difficulty, this study assumes that the active constraint set, which is identified at the beginning of each inner loop iteration, stays fixed during an inner loop iteration. In other words, when solving the FMDUE (or adjusting flows on existing paths) in the inner loop, we assume the shortest paths are fixed in an inner loop iteration l . That is, $GC_{odp}^r(b, r) = \min\{GC_{odp}^r(b, r), \forall p \in P(b, o, d, \tau)\}$. This active constraint set strategy had also been applied in several DTA studies, such as Huang and Lam (2002), and Szeto and Lo (2005).

This work proposes a multi-class path-swapping descent direction method that is able to circumvent the need to calculate partial derivatives and to optimally determine step sizes, while, in the mean time, maintains the mechanisms of searching along (feasible) descent directions. The proposed heuristic decomposes the NMP into many (b, o, d, τ) sub-problems and solves each of them by adjusting time-varying, multi-class O-D flows between non-least generalized cost paths and the (referenced) least generalized cost path. Given a feasible solution $r^l \in \Omega$ in an inner loop iteration l , the method features the following form:

$$r^{l+1} = P_{\Omega}[r^l - \rho^l \times Dir^l] = P_{\Omega}[r^l - \rho^l \times \frac{r^l \times (GC(r^l) - GC^*(r^l))}{GC(r^l)}] \quad (15)$$

where $-Dir^l$ is the (feasible) descent direction and $\rho^l \in (0, 1)$ is the step size in iteration l ; $P_{\Omega}[u]$ denotes the unique projection of path flow vector u onto Ω (the set of feasible path flow vectors) and is defined as the unique solution of the problem: $\min_{v \in \Omega} \|u - v\|$. $GC(r^l) \equiv \{GC_{odp}^r(b, r^l)$,

$\forall b, o, d, \tau, p \in P(b, o, d, \tau)$; $GC^*(r^l) \equiv \{GC_{odp^*}^\tau(b, r^l), \forall b, o, d, \tau, p \in P(b, o, d, \tau)\}$ is the multi-class least path generalized cost vector evaluated at a flow pattern r^l .

Proposition 2: ($-Dir^l$) is the feasible descent searching direction corresponding to the iterate r^l .

Proof: the proof of Proposition 2 is given in the Appendix.

According to Eq.(15), the new iterate r^{l+1} is obtained by updating the current iterate r^l along the direction $-Dir^l$ with a move size ρ^l . Specifically, for each (b, o, d, τ) sub-problem, the proposed multi-class path-swapping descent direction method updates the current path assignment r^l as follows:

$$r_{odp}^{\tau, l+1}(b) = \max\{0, r_{odp}^{\tau, l}(b) - \rho^l \times \frac{r_{odp}^{\tau, l}(b) \times [GC_{odp}^\tau(b, r^l) - GC_{odp^*}^\tau(b, r^l)]}{GC_{odp}^\tau(b, r^l)}\},$$

$$\forall p \in P(b, o, d, \tau), p \neq p^*; \quad (16a)$$

$$r_{odp^*}^{\tau, l+1}(b) = r_{odp^*}^{\tau, l}(b) + \sum_{p \in P(b, o, d, \tau), p \neq p^*} \rho^l \times \frac{r_{odp}^{\tau, l}(b) \times [GC_{odp}^\tau(b, r^l) - GC_{odp^*}^\tau(b, r^l)]}{GC_{odp}^\tau(b, r^l)}, \quad (16b)$$

where step size ρ^l is determined by the scheme of mixed step sizes, described in the following

$$\rho^l = 1/k, \text{ if } l = 0; \rho^l = 1, \text{ otherwise.} \quad (17)$$

This multi-class path assignment updating scheme is intuitively based on the fact that travelers farther from the equilibrium and on paths with larger flow rates are more inclined to change path than those on paths with smaller flow rates and with travel cost closer to the minimal cost.

5. NUMERICAL EXPERIMENTS

A set of numerical experiments is conducted to compare the solution quality of the proposed IMDUE algorithm (termed CG for column generation) and that of the MSA-based algorithm, developed by Lu et al. (2008), in addition to examining the convergence pattern and solution quality of the new algorithm. Both algorithms are coded and compiled by using the Compaq Visual FORTRAN 6.6 and evaluated on the Windows XP platform and a machine with an Intel Pentium IV 2.8 GHz CPU and 2GB RAM.

In all the experiments conducted, the following parameter settings are applied. The continuous VOT distribution considered in the experiments is a normal distribution with (mean, standard deviation) = (24, 12), denoted as N(24, 12). The parameters of this normal distribution are adapted from the estimated measurements in a value pricing experiment conducted in Southern California, USA (Brownstone and Small, 2005), and the unit of VOT in this study is United States dollars (USD) per hour. The feasible range of the VOT distribution $[\alpha^{\min}, \alpha^{\max}]$ is [0.6, 180].

A strict convergence criterion is used in the inner loop of the column generation-based algorithm; that is $|Gap(r^l) - Gap(r^{l-1})|/Gap(r^l) \leq 0.001$. The initial solutions are obtained by loading time-varying O-D demands to the extreme non-dominated paths calculated based on

prevailing travel times output from the traffic simulator. Another measure of effectiveness (MOE), $AGap(r)$, is also collected in the conducted experiments, in addition to $Gap(r)$.

$$AGap(r) = \frac{\sum_b \sum_o \sum_d \sum_\tau \sum_{p \in P(b,o,d,\tau)} r_{odp}^\tau(b) \times [GC_{odp}^\tau(b,r) - GC_{odp^*}^\tau(b,r)]}{\sum_b \sum_o \sum_d \sum_\tau \sum_{p \in P(b,o,d,\tau)} r_{odp}^\tau(b)} \quad (18)$$

This MOE, which is the average gap over all vehicles in the network for a given path flow pattern r , is independent of problem sizes and thus useful for examining the convergence pattern and solution quality of the proposed algorithm on different networks. The minimum of the $AGap(r)$ is zero. Essentially, the smaller the average gap, the closer the solution is to the IMDUE.

5.1 Experiments on Irvine network

The Irvine (California, USA) network consists of 326 nodes (70 of them are signalized), 626 links, and 61 traffic analysis zones (TAZ). This network had been calibrated by using real-world observations from multiple-day detector data (Mahmassani et al., 2003). A 2-hour (7-9AM) morning peak time-varying O-D demand table is extracted from a 6-hour (4-10AM) demand table and loaded to the test network, with 35,300 vehicles in the observation period (7:10-8:50AM). To create hypothetical dynamic pricing scenarios, one lane of a portion (about 1 mile) of the I-405 westbound freeway is converted to a toll road, along with an additional new toll lane. The two toll lanes have the same length as the (remaining) three regular lanes but a 10-mile higher posted speed limit (and hence higher capacity) than the regular lanes. Table 1 lists the three simple dynamic pricing scenarios tested in the experiment conducted on the Irvine network. These three pricing scenarios have the same four pricing periods but different toll levels, each representing low, middle, and high toll scenarios, respectively.

Table 1 Dynamic road pricing scenarios tested on Irvine network

Pricing Scenario	Period 1 (7:00-7:30AM)	Period 2 (7:30-8:00AM)	Period 3 (8:00-8:30AM)	Period 4 (8:30-9:00AM)
1 (Low)	\$0.10	\$0.20	\$0.30	\$0.15
2 (Middle)	\$0.20	\$0.30	\$0.40	\$0.25
3 (High)	\$0.30	\$0.40	\$0.50	\$0.35

The convergence patterns in terms of iteration-by-iteration gap values of the CG algorithm under the three dynamic pricing scenarios are presented in Table 2. It can be seen that the algorithm can effectively reduce the gap measure as well as the average gap in all three pricing scenarios tested on the Irvine network, although the convergence patterns are not strictly monotonic decreasing. As for the solution quality, the final gap values obtained by the new algorithm are 3.9% (196.3/5028.6), 4.5% (234.9/5211.2), and 5.4% (315.1/5795.7) of the initial gap values, respectively, for the three pricing scenarios. In addition, the average gap values for the three pricing scenarios, obtained by dividing these final gap values by the number of vehicles loaded in the observation period, are all less than 0.01 minutes. These small gap and average gap values indicate that the proposed algorithm is able to find close-to-IMDUE solutions for this network application.

Table 2 Convergence patterns of the CG algorithm on Irvine network

Iteration	Gap(r)			AGap(r)		
	Scenario 1	Scenario 2	Scenario 3	Scenario 1	Scenario 2	Scenario 3
0	5028.6	5211.2	5795.7	0.142	0.148	0.164
1	835.0	1025.6	851.3	0.024	0.029	0.024
2	787.1	892.2	822.7	0.022	0.025	0.023
3	452.8	624.6	546.9	0.013	0.018	0.015
4	536.9	505.0	501.4	0.015	0.014	0.014
5	590.7	597.7	407.3	0.017	0.017	0.012
6	376.1	415.4	542.2	0.011	0.012	0.015
7	409.6	332.2	419.5	0.012	0.009	0.012
8	523.4	342.0	385.8	0.015	0.010	0.011
9	316.2	369.4	366.9	0.009	0.010	0.010
10	406.5	357.9	299.1	0.012	0.010	0.008
11	372.9	280.1	460.6	0.011	0.008	0.013
12	430.7	294.8	402.2	0.012	0.008	0.011
13	335.7	238.9	237.7	0.010	0.007	0.007
14	589.1	256.4	292.6	0.017	0.007	0.008
15	274.5	255.4	320.2	0.008	0.007	0.009
16	283.4	252.9	353.9	0.008	0.007	0.010
17	271.2	228.3	249.3	0.008	0.006	0.007
18	247.1	268.3	323.7	0.007	0.008	0.009
19	258.4	285.3	313.0	0.007	0.008	0.009
20	196.3	234.9	315.1	0.006	0.007	0.009

The comparison of solution quality of CG and MSA is reported in Table 3 and Fig. 2. As shown in the figure and table, the proposed algorithm, CG, obtains a much better solution in terms of $Gap(r)$ and $AGap(r)$. The final $AGap(r)$ value obtained by MSA (0.056 min) is 8 times larger than that obtained by CG (0.007 min). The gap reduction percentages of CG and MSA are 96% $((5211.2-234.9) / 5211.2)$ and 62% $((5211.2-1985.1) / 5211.2)$, respectively. The computation times of CG and MSA on Irvine network are about 24 and 23 hours, respectively. Thus, CG reduces the initial gap value 34% (=96%–62%) more than MSA with just 4.3% more of computation time.

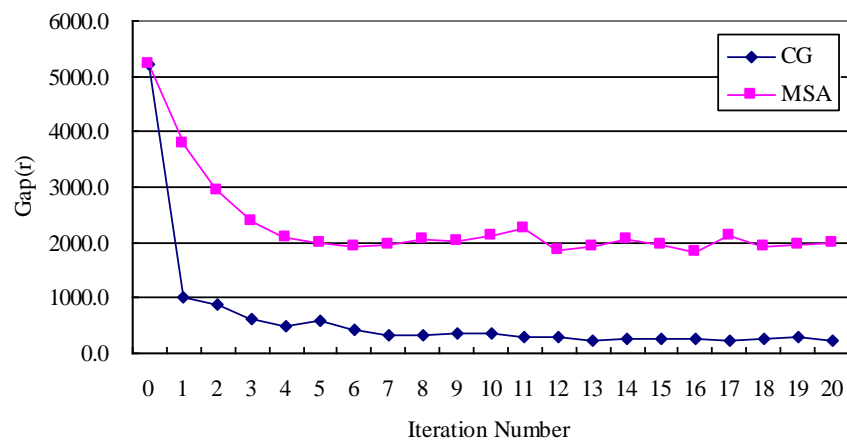


Fig. 2 Comparison of solution quality on Irvine network

Table 3 Comparison of solution quality on Irvine network

Iteration	CG		MSA	
	<i>Gap(r)</i>	<i>AGap(r)</i>	<i>Gap(r)</i>	<i>AGap(r)</i>
0	5211.2	0.148	5211.2	0.148
1	1025.6	0.029	3781.3	0.107
2	892.2	0.025	2925.5	0.083
3	624.6	0.018	2368.5	0.067
4	505.0	0.014	2102.6	0.060
5	597.7	0.017	1985.7	0.056
6	415.4	0.012	1934.1	0.055
7	332.2	0.009	1972.3	0.056
8	342.0	0.010	2053.4	0.058
9	369.4	0.010	2029.5	0.057
10	357.9	0.010	2130.6	0.060
11	280.1	0.008	2244.4	0.064
12	294.8	0.008	1851.9	0.052
13	238.9	0.007	1923.5	0.054
14	256.4	0.007	2045.4	0.058
15	255.4	0.007	1959.2	0.056
16	252.9	0.007	1831.8	0.052
17	228.3	0.006	2117.5	0.060
18	268.3	0.008	1921.0	0.054
19	285.3	0.008	1952.3	0.055
20	234.9	0.007	1985.1	0.056

5.2 Experiments on CHART network

To further demonstrate the capability of the proposed algorithm for large-scale networks with dynamic road pricing scenarios, another experiment is conducted on a large road network: the Maryland CHART network, which consists primarily of the I-95 freeway corridor between Washington, D.C. and Baltimore (Maryland, USA) and is bounded by two beltways (I-695 Baltimore Beltway on the north and I-495 Capital Beltway on the south). The CHART network has 2241 nodes (231 of them are signalized), 3459 links and 111 traffic analysis zones (TAZ). This network had been calibrated by using real-world observations from multiple-day detector data (Mahmassani et al. 2006). An available 1-hour (7:30-8:30AM) morning peak time-varying O-D demand (with 39,560 vehicles in the observation period from 7:40 to 8:20 AM) table is extracted and loaded to the network. To create hypothetic dynamic toll scenarios, one of the 20-mile long southbound lanes of the I-95 corridor is converted to the toll road, together with an additional new toll lane. The two toll lanes have the same length, posted speed limit, and capacity as the (remaining) three regular lanes. The two-lane toll road consists of 57 links in the coded network, and the four access/egress points to/from the toll road are interchanges with I-195, MD-100, MD-32 and MD-198, where additional on-ramps and off-ramps are added. A dynamic link toll vector generated by the method proposed by Dong et al. (2011) is used in this network to test the BDUE algorithm.

The comparison of solution quality of CG and MSA is reported in Table 4 and Fig. 3. As presented in the figure and table, CG significantly outperforms MSA, because the final *AGap(r)* value obtained by MSA (0.149 min) is almost 19 times larger than that obtained by CG (0.008 min). The gap reduction percentages of CG and MSA are 98.8% $((25393.2-300.1) / 25393.2)$ and

76.7% $((25393.2-5915.4) / 25393.2)$, respectively. The computation times of CG and MSA on CHART network are about 40 and 39 hours, respectively. Similar to the result found on Irvine network, CG reduces the initial gap value 22% $(=98.8\%-76.7\%)$ more than MSA with just 2.5% more of computation time. While MSA was reported in the previous work (e.g., Lu et al., 2008) to be able to find acceptable solutions, the results on both networks shown in the current paper demonstrate that CG is more effective than MSA in obtaining close-to-IMDUE solutions. Note that for planning purpose, those amounts of computation time are acceptable, and less computation time can be achieved with more powerful machines and/or more efficient implementation (coding) of the algorithm.

Table 4 Comparison of solution quality on CHART network

Iteration	CG		MSA	
	<i>Gap(r)</i>	<i>AGap(r)</i>	<i>Gap(r)</i>	<i>AGap(r)</i>
0	25393.2	0.641	25393.2	0.641
1	5622.9	0.142	11082.9	0.280
2	3207.4	0.081	13210.4	0.334
3	1885.5	0.048	11933.5	0.301
4	1379.7	0.035	9847.8	0.249
5	1051.3	0.027	8325.4	0.210
6	984.0	0.025	7775.6	0.196
7	814.4	0.021	6397.9	0.162
8	816.0	0.021	6072.8	0.153
9	621.6	0.016	6591.1	0.166
10	642.8	0.016	7178.3	0.181
11	492.1	0.012	6325.7	0.160
12	427.8	0.011	5637.8	0.142
13	415.8	0.011	6104.7	0.154
14	310.2	0.008	6389.6	0.161
15	300.1	0.008	5915.4	0.149

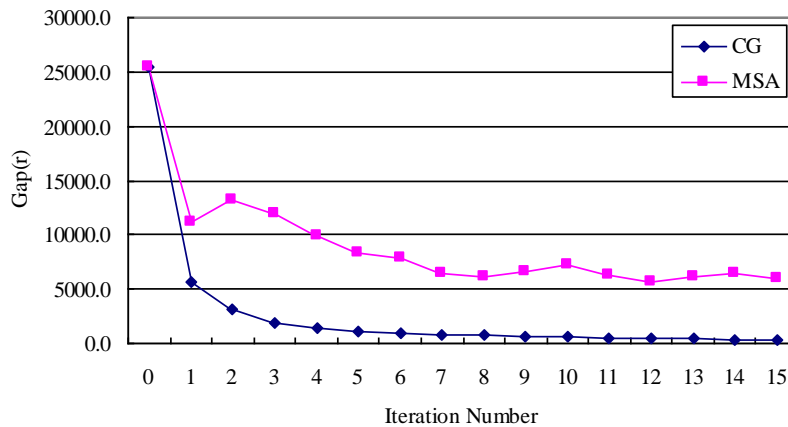


Fig. 3 Comparison of solution quality on CHART network

6 Concluding Remarks

This paper presents an efficient algorithm for solving the IMDUE problem, or specifically the BDUE problem, which assumes the VOT is continuously distributed across the population of trips. The proposed column generation-based algorithm (i) applies the parametric analysis method in the outer loop to determine multiple user classes and to generate representative extreme non-dominated paths, and (ii) solves FMDUE problems in the inner loop by a feasible descent direction method. Although the mathematical abstraction of the problem is a typical analytical formulation, the solution algorithm adopts the simulation-based approach to tackle many practical aspects of the DTA applications. The experimental results show that the convergence pattern of the proposed algorithm is not affected by the different VOT assumptions (constant or random VOT), and the algorithm is able to find close-to-IMDUE (or approximate) solutions. In addition, the solution quality of the proposed algorithm is much better than that of the MSA-based solution method.

Several interesting research directions can be continued based on the rich modeling capabilities of the IMDUE model in capturing traffic dynamics and user heterogeneity. For instance, the model can be extended to consider O-D-specific and/or time-varying VOT distributions, provided that the data are available to estimate the underlying parameter distributions. The model can also be integrated into a solution framework aiming at finding optimal or Pareto-improving dynamic pricing schemes, including locations, pricing periods and toll charges, so as to alleviate congestion. In addition, incorporating stochastic path choice with explicit perception errors (e.g. logit or probit models) would be an important and interesting extension. It will be interesting to implement some other VI algorithms developed for the static traffic assignment problems and compare their performance with that of our proposed approach. The model presented in this paper can be viewed as laying the foundation for a platform that integrates more realistic behavioral modeling in a dynamic network analysis tool. The main challenges in this development is to continue pushing the boundary of what can be realistically handled in a large network setting within the limits of practical computational capabilities.

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APPENDIX

Proof of Proposition 1:

Suppose r^* satisfies the MDUE condition, and let $G(r^*)$ be the corresponding path generalized cost vector. According to the MDUE definition, for an equilibrium multi-class path flow vector r^* , the following condition can be established:

If $G_{odq}^\tau(\alpha, r^*) > G_{odp}^\tau(\alpha, r^*) (= \pi_{od}^\tau(\alpha))$, then $r_{odq}^\tau(\alpha) = 0$.

If $G_{odq}^\tau(\alpha, r^*) = G_{odp}^\tau(\alpha, r^*) (= \pi_{od}^\tau(\alpha))$, then $r_{odq}^\tau(\alpha) \geq 0$.

$$\forall o, d, \tau, p \text{ and } q \in P(o, d, \tau), \text{ and } \forall \alpha \in [\alpha^{\min}, \alpha^{\max}] \quad (\text{A1})$$

where $\pi_{od}^\tau(\alpha)$ is the minimum possible generalized travel cost for the trips with VOT α from o to d departing at time τ .

Consider these path generalized costs $G(r^*)$ as fixed at the current level of path flow r^* . Because r^* satisfies the MDUE condition and hence only least generalized cost paths are used, the total generalized cost cannot be reduced by moving flows from least generalized cost paths to other inefficient paths. For instance, if path generalized costs are fixed as $G(r^*)$ and $G_{odq}^\tau(\alpha, r^*) > G_{odp}^\tau(\alpha, r^*)$ then moving flow $r_{odp}^\tau(\alpha)$ from p to q will lead to an increase of total generalized cost by $r_{odp}^\tau(\alpha) \times G_{odq}^\tau(\alpha, r^*) - r_{odp}^\tau(\alpha) \times G_{odp}^\tau(\alpha, r^*) > 0$. Therefore any other feasible multi-class path flow vector $r \in \Omega$ has total generalized cost at least as large as r^* which uses only cheapest paths. In other words, the MDUE path flow vector r^* satisfying (A1) is also the solution of the infinite dimensional variational inequality (8):

$$G(r^*)^T \circ r^* \leq G(r^*)^T \circ r \quad \text{or} \quad G(r^*)^T \circ (r^* - r) \leq 0, \quad \forall r \in \Omega \quad (\text{A2})$$

Conversely, assume that condition (A1) does not hold, then there exists the following situation for some triplet (o, d, τ) and paths p and q : $r_{odq}^\tau(\alpha) > 0$ and $G_{odq}^\tau(\alpha, r^*) > G_{odp}^\tau(\alpha, r^*) (= \pi_{od}^\tau(\alpha))$. Moving flow $r_{odq}^\tau(\alpha)$ from q to the cheaper path p will result in a reduction of total generalized cost by $r_{odq}^\tau(\alpha) \times G_{odq}^\tau(\alpha, r^*) - r_{odq}^\tau(\alpha) \times G_{odp}^\tau(\alpha, r^*) > 0$. Let the resulting flow pattern be $r \in \Omega$. Then $G(r^*)^T \circ r < G(r^*)^T \circ r^*$, in which case Eq.(8) is not satisfied.

From above, if (A1) is satisfied then Eq.(8) is satisfied, and if (A1) does not hold then Eq.(8) does not, either. Thus, conditions (A1) and Eq.(8) are equivalent, and solving for the IMDUE flow pattern is equivalent to finding the solution of the (possibly) infinite dimensional variational inequality Eq.(8). This completes the proof.

Proof of Proposition 2:

Denote by ϕ_a^t the time-dependent link marginal travel time: the travel time contribution of an additional unit of vehicular flow on link a in time interval t to the link travel time d_a^t . By assuming that d_a^t is a monotonic (increasing) function of x_a^t (the number of vehicles on link a in time interval t): $[(x_a^t + \Delta x_a^t) - x_a^t] \times [d_a^t(x_a^t + \Delta x_a^t) - d_a^t(x_a^t)] \geq 0$ (e.g. Nagurney, 1998), with $\Delta x_a^t > 0$, the following can be obtained:

$$\phi_a^t = \lim_{\Delta x_a^t \rightarrow 0} \frac{d_a^t(x_a^t + \Delta x_a^t) - d_a^t(x_a^t)}{\Delta x_a^t} \geq 0. \quad (\text{A3})$$

Note that this study considers ϕ_a^t as a local link marginal. Peeta (1994) gave a comprehensive discussion on global link marginals with temporal and spatial interactions.

Since link costs (i.e. tolls) c_a^t are given as input, the path marginal generalized cost perceived by the tripmaker of user class $u(b)$, $\phi_{odp}^\tau(b, r)$, is assumed to be the sum of constituent link marginal travel times weighted by the VOT of user class $u(b)$.

$$\eta_{odp}^\tau(b, r) = \sum_{a \in A(p)} \phi_a^t \times \delta_{odpa}^{\tau, t} \times \alpha(b) \quad (\text{A4})$$

where $A(p)$ is the set of links on path p , t is the first time interval in which link a on path p is reached by a vehicle assigned to that path in time τ , and $\delta_{odpa}^{\tau, t}$ is the time-dependent link-path incidence indicator; $\delta_{odpa}^{\tau, t} = 1$ if vehicles going from o to d assigned to path p at time τ pass link a in time interval t , and 0 otherwise.

Recall that p^* be the referenced shortest path for a (b, o, d, τ) . Then constraints (14c) can be re-written as the following:

$$r_{odp^*}^\tau(b) = h_{od}^\tau(b) - \sum_{p \in P(b, o, d, \tau) \setminus p^*} r_{odp}^\tau(b), \forall b, o, d, \text{ and } \tau. \quad (\text{A5})$$

Define a new path flow vector $y = \{y_{odp}^\tau(b), \forall p \in P(b, o, d, \tau) \setminus p^*, \forall b, o, d, \tau\}$. By substituting Eq.(A3) into the objective function (14a), the NMP becomes the following unconstrained minimization problem:

$$\text{Min Gap}(y) = \sum_b \sum_o \sum_d \sum_\tau \sum_{p \in P(b, o, d, \tau) \setminus p^*} y_{odp}^\tau(b) \times [GC_{odp}^\tau(b, y) - GC_{odp^*}^\tau(b, y)] \quad (\text{A6})$$

Note that the constraints (14b) and (14d) are satisfied in the NMP because of the aforementioned active constraint set strategy and the projection of the updated solution onto the feasible set Ω , respectively. With this transformation and according to Eq.(A4), the first-order partial derivative of $\text{Gap}(r)$ with respect to a particular $y_{odp}^\tau(b)$ is obtained as the following:

$$\begin{aligned} \frac{\partial \text{Gap}(y)}{\partial y_{odp}^\tau(b)} &= GC_{odp}^\tau(b, y) - GC_{odp^*}^\tau(b, y) + y_{odp}^\tau(b) \times \frac{\partial [GC_{odp}^\tau(b, y) - GC_{odp^*}^\tau(b, y)]}{\partial y_{odp}^\tau(b)} \\ &+ \sum_{p' \in P(b, o, d, \tau) \setminus (p^* \cup p)} y_{odp'}^\tau(b) \times \frac{\partial [GC_{odp'}^\tau(b, y) - GC_{odp^*}^\tau(b, y)]}{\partial y_{odp}^\tau(b)} \\ &= GC_{odp}^\tau(b, y) - GC_{odp^*}^\tau(b, y) + y_{odp}^\tau(b) \times \sum_{a \in B(p)} \phi_a^t \times \alpha(b) \\ &+ \sum_{p' \in P(b, o, d, \tau) \setminus (p^* \cup p)} (y_{odp'}^\tau(b) \times \sum_{a \in A(p') \cap B(p)} \phi_a^t \times \alpha(b)) \end{aligned} \quad (\text{A7})$$

where $A(p)$ is the set of links on path p , and $B(p) = \overline{A(p) \cap A(p^*)}$ is the set of links that are on either the non-shortest path p or the referenced shortest path p^* . In the following, we prove that the search direction $[y \frac{GC(y) - GC^*(y)}{GC(y)}]$ is a descent direction of $Gap(y)$ at y .

To prove the vector $Dir = [y \frac{GC(y) - GC^*(y)}{GC(y)}]$ is a descent direction of $Gap(y)$ at y , it is

necessary to show that the inner product $\nabla Gap(y) \bullet (-1 \times Dir) < 0$ (see e.g. Theorem 4.1.2 in Bazara'a et al. 1993). Component-wise, this is equivalent to showing that

$$\sum_{p \in P(b, o, d, \tau) \setminus p^*} ((-1) \times \frac{\partial Gap(y)}{\partial y_{odp}^\tau}(b) \times Dir_{odp}^\tau(b)) < 0, \forall b, o, d, \text{ and } \tau, \quad (A8)$$

where $Dir_{odp}^\tau(b) = [y_{odp}^\tau(b) \times \frac{(GC_{odp}^\tau(b, y) - GC_{odp^*}^\tau(b, y))}{GC_{odp}^\tau(b, y)}]$ and $\frac{\partial Gap(y)}{\partial y_{odp}^\tau}(b)$ is defined as Eq.(A7).

Consider that, for a (b, o, d, τ) and for each path p ($r_{odp}^\tau(b) > 0$) in the path set $P(b, o, d, \tau) \setminus p^*$, the cost of path p could be either equal to or greater than the least cost. In the first case, p is one of the shortest (more precisely, least cost) paths, then $Dir_{odp}^\tau(b) = 0$ and accordingly $(-1) \times \frac{\partial Gap(y)}{\partial y_{odp}^\tau}(b) \times Dir_{odp}^\tau(b) = 0$. In the latter case, p is a non-shortest path, then $Dir_{odp}^\tau(b) > 0$.

According to Eq.(A3), link marginal travel times are non-negative and $GC_{odp}^\tau(b, y) - GC_{odp^*}^\tau(b, y)$ is positive for any non-shortest path p , so $\frac{\partial Gap(y)}{\partial y_{odp}^\tau}(b) > 0$ and $(-1) \times \frac{\partial Gap(y)}{\partial y_{odp}^\tau}(b) \times Dir_{odp}^\tau(b) < 0$.

Mathematically, for each (b, o, d, τ)

$$\begin{aligned} & \sum_{p \in P(b, o, d, \tau) \setminus p^*} [(-1) \times \frac{\partial Gap(y)}{\partial y_{odp}^\tau}(b) \times Dir_{odp}^\tau(b)] \\ &= - \sum_{p \in P_\pi(b, o, d, \tau) \setminus p^*} [\frac{\partial Gap(y)}{\partial y_{odp}^\tau}(b) \times Dir_{odp}^\tau(b)] - \sum_{p \in P(b, o, d, \tau) \setminus P_\pi(b, o, d, \tau)} [\frac{\partial Gap(y)}{\partial y_{odp}^\tau}(b) \times Dir_{odp}^\tau(b)] \quad (A9) \\ &= 0 - \sum_{p \in P(b, o, d, \tau) \setminus P_\pi(b, o, d, \tau)} [\frac{\partial Gap(y)}{\partial y_{odp}^\tau}(b) \times Dir_{odp}^\tau(b)] < 0 \end{aligned}$$

where $P_\pi(b, o, d, \tau)$ is the set of least generalized cost paths of a (b, o, d, τ) . Thus the search direction Dir is a descent direction of $Gap(y)$ at y . This completes the proof.