OPTIMAL COMMUTING AND WORK START TIME DISTRIBUTION UNDER FLEXIBLE WORK HOURS SYSTEM ON MOTOR COMMUTING

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Abstract: Recently, the "Flexible work hours (FWH) system" is focused as a means to level the peak of commuting demand. However, for business activities subject to temporal agglomeration effect, the FWH system decreases the number of concurrently working people, and so the productivity is decreased, even though all workers still work for the same hours as before. This paper formulates an optimal control problem under the FWH system on motor commuting, considering one-day schedule and temporal agglomeration effect. We show that the one of the optimal distribution patterns for work start time is, some commuters start at fixed time and others start continuously before or after that fixed time. As a result, we make it clear, how could the FWH be introduced into firms to realize the optimal pattern. Furthermore, this model can calculate the social effect of the FWH.

Key Words: Work Start Time Choice, Temporal Agglomeration Economies, Optimal Control

1. INTRODUCTION

In almost all Asian metropolises, as road infrastructure is far below from the satisfaction level, we are suffering from heavy road congestion and environmental problem due to exhaust gas. Considering both monetary and space constraints, we cannot expect large amount of road facilities expansion, any more. Therefore, the "Flexible Work Hours (FWH) system" is introduced recently in order to reduce the peak of commuting demand. FWH is one of the transportation demand management measures to change the distribution of home departure time, as well as that of the work start time. However, for business activities, FWH decreases temporal agglomeration effect on productivity, because this staggers the work start time. Thus, each commuter chooses his / her home departure time and work start time considering the tradeoff between commuting disutility that consists of congestion and schedule cost, and income differentiated by work start time.

A lot of studies analyzed theoretically this road congestion problem and the FWH till now. However, these studies are not concerned with one-day schedule including the return in the evening or difference of productivity for each work start time.

This paper formulates an optimal control problem with the FWH system under user equilibrium situation and system optimal situation on motor commuting. Commuter road has one bottleneck, and firm's productivity in a metropolitan area is subject to temporal agglomeration effect. The purpose of this study is to solve this optimal problem, and to make some feasible patterns for home departure / return time and work start (end) time distributions theoretically. Furthermore, in this paper we have calculated social utility value for each commuting pattern.

Specifically, we have formulated firm's productivity, schedule cost and traffic congestion, and the model can decide home departure / return time and work start (end) time distributions as endogenous. As a result, we show the FWH introduction rate, which enable the optimal pattern. The model can also calculate social utility value for each pattern. We compared the utility value under the FWH with that under the fixed work start time. Besides, the economic effects of the FWH have also been identified in this paper.

2. RELATED STUDIES UP TO NOW

One of the easiest ways to describe the relationship between the demand and trip time along congested road was queuing theory. A pioneering study utilized queuing theory to describe waiting time at one bottleneck (Vickrey, 1969), and it was expanded for road network consist of many links (Filipiak, 1981). When commuters choose their home departure time, they meet with tradeoff between travel time and schedule cost, which is related to the difference between desired time and actual departure time. At first, deterministic models, describing such tradeoff, were developed to investigate the existence or uniqueness of equilibrium (Hendrickson, C. *et al.*,1981). Remarkable development of logit models in 1980's enabled reserchers to build stochastic models and simultaneous models including route choice as well as time selection. Most of these models divide time axis to several number of periods and formulate discrete choice models. Continuous optimal condition along time axis were also investigated by optimal control theory (de Palma, A. *et al.*,1983, Arnott, R. *et al.*,1993). Arnott, R. *et al.*(1990) used deterministic model, and compared the social optimum and no-toll equilibrium. Furthermore, those were extended to the situation of two different types of commuters in desired work start time (Arnott, R. *et al.*,1998), and stochastic capacity and demand in the bottleneck (Arnott, R. *et al.*,1999).

It was late 1960's when the people came to realize that expansion of transportation facilities couldn't solve the transportation congestion problem considering the enormous time that it would take and the forbidden cost for expansion. Then, they began to focus transportation demand management (TDM) measures such as changing home departure time distribution in order to level the peak of demand.

In Japan, from more practical viewpoint, social experiments of TDM measures such as staggered commuting became reported in 1990's. Some of them report TDM measures have effect to lessen traffic congestion. But, on the other hand, strong reluctance of firms to join the staggered work hours and / or flexible work hours, and explanation of its negative effect on business efficiency were also reported. Business activities locate in urban area because they expect better accessibility to other activities and more effective interactions with others. Hall(1991) insisted that such effect also exists on the time axis and call it by the term of "temporal agglomeration economy". When you happen to need urgent communication with some person in other firm, if the person is not on work, you must wait him / her. FWH have the possibility to increase such temporal mismatches and harm the temporal agglomeration effect.

The effect of "temporal agglomeration economy" in commuting with FWH system was first analyzed by Henderson(1977, 1981). He theoretically proves that firms are subject to temporal agglomeration economies, and their productive effect is reduced by the staggered work hours in which each firm starts working at different time. However, his model predicts the equilibrium situation where work start time of firms is distributed continuously. Mun, S. *et al.*(2000) analyzed firm's incentive to FWH system considering temporal agglomeration. They compared the firm's utility under the fixed work start time with that under FWH, and consequently, they investigated the incentive of firm to introduce the FWH.

If some people change their work start time, they can change their work end time, and they may return home early, accordingly. Furthermore, temporal agglomeration economy also effects in the evening, by reducing their productivity, if some people end their work earlier. However, all these studies are not concerned with one-day schedule including the return in the evening.

3. FORMURATOIN OF THE MODEL

3.1 Problem Settings of Our Study

We assume a single road connecting a residential area and the CBD, which has one bottleneck just before the CBD. All of N commuters drive cars to CBD where they have their jobs. No other transportation mode can be used for commuting, therefore, daily demand is fixed as N at the bottleneck. It is assumed that vehicles are driven at constant speed from home to just before the bottleneck, and time for this section is equal to constant w regardless of housing location of the commuters. From this assumption, we can use arrival time distribution at the bottleneck instead of home departure time distribution. Queue occurs when traffic is over bottleneck's capacity k (vehicle / min).

All firms in the city are located at the CBD and have the same technology of production. All workers work just H minutes a day. The firms can specify the working schedules (i.e. work start time and work end time) for some group of workers, or introduce FWH system and let each worker choose his / her working schedule by him / herself. In latter case, the firms offer to each of the workers different wage rate based on the productivity of their work schedules.

3.2 Specification of Commuters Behavior

Utility levels of a representative commuter, q, in the morning / evening are defined as the following functions.

$$U(q) = \{-e_1(m(q)-a(q))\} + \{-c_1(T_1 - m(q))\} \tag{1}$$
$$V(q) = \{-e_2(b(q)-l(q))\} + \{-c_2(l(q) - T_2)\} \tag{2}$$

where, $U(q)$: morning part of disutility, $V(q)$: evening part of disutility. $a(q)$: home departure time, $m(q)$: office arrival time, $l(q)$: office leaving time, $b(q)$: return home time. T_1, T_2 : arbitrary origin point in time axis, $e_1(> e_2)$, $e_2(> 0)$: slope of queue cost, $c_1(> c_2, < e_1)$, $c_2(> 0, < e_2)$: slope of schedule cost. In these equations the first term represents the disutility of queue time, second term represents the schedule cost for early departure in the morning / late return home in the evening.

Whenever queue occurs at the bottleneck, traffic outflow rate equals k. Then, considering if this rate is smaller than k, their schedule cost increases along time. Office arrival time $m(q)$ and home arrival time $b(q)$ are represented as follows;

$$m(q) = \frac{q}{k} + S_0 \tag{3}$$
$$b(q) = \frac{q}{k} + (S_1 + H) \tag{4}$$

where, S_0 : office arrival time of the first commuter, S_1 : work start time of the first commuter.

Firms locating at the CBD utilize the labor force as the single input, and produce numeraire goods. All firms have the same technology and are subjected to the temporal agglomeration effect. From the free entry condition, labor market becomes perfectly competitive. Then each produced value is equally divided among each worker. We formulate it by using instant production function like Henderson(1977, 1981) as follows;

$$Y(q) = \int_{t_s}^{t_s+H} A\rho(\tau)^{\alpha} d\tau \tag{5}$$

where, $Y(q)$: wage rate of commuter q, A : technology level parameter, $\rho(\tau)$: number of the labor at work in the city at time τ, t_s : work start time of commuter q. α is temporal agglomeration economy parameter, large for the manufacture industry such as automobile manufactures, and small for the firms with intellectual works such as consultants and academic institutes.

Now, let us define $n(q)$ as the work start time of q^{th} worker. If work hour H is long enough that there is an instant when all workers are at work, then $\rho(t)$ can be calculated by;

$$\rho(t) = \begin{cases} Bn^{-1}(t) & \text{if } \ t \in [S_1, S_2) \\ BN & \text{if } \ t \in [S_2, S_1 + H] \\ B\left(N - n^{-1}(t-H)\right) & \text{if } \ t \in (S_1 + H, S_2 + H] \end{cases} \tag{6}$$

where, B : number of roads connected to the CBD which are independent of others and have same bottleneck capacity, $n^{-1}(t)(=q)$: inverse function of $n(q)$. S_1 : work start time of the first worker, S_2 : work start time of the last worker, $S_1 + H$: work end time of the first worker, $S_2 + H$: work end time of the last worker.

As a result, the following equation gives the wage rate of q^{th} worker, $Y(q, S_1, S_2)$;

$$\begin{aligned} Y(q, S_1, S_2) &= \int_{n(q)}^{S_2} A(Bn^{-1}(\tau))^{\alpha} d\tau + AB^{\alpha}N^{\alpha}(S_1 + H - S_2) \\ &\quad + \int_{S_1}^{n(q)} A\left\{B\left(N - n^{-1}(\tau)\right)\right\}^{\alpha} d\tau \end{aligned} \tag{7}$$

Utility level of q is finally defined as the following function;

$$W(q) = Y(q) + U(q) + V(q) \tag{8}$$

And, social welfare in this system is defined as the following function which is integrated commuter's utility $W(q)$ from 0 to N;

$$\begin{aligned} SW &= \int_0^N W(q)dq \\ &= \int_0^N \Bigg[Y(q, S_1, S_2) - e_1\left\{\left(\frac{q}{k} + S_0\right) - a(q)\right\} - c_1\left\{T_1 - \left(\frac{q}{k} + S_0\right)\right\} \\ &\quad - e_2\left\{\left(\frac{q}{k} + (S_1 + H)\right) - l(q)\right\} - c_2\{l(q) - T_2\}\Bigg] dq \end{aligned} \tag{9}$$

3.3 User Equilibrium Model under FWH (Second Best)

If workers are told to choose each schedule times, i.e. $a(q)$: home departure time , $n(q)$: work start (end) time, and $l(q)$: office leaving time, as they like without any political intervention, the distributions of such schedule would be steady on an equilibrium situation, where none can expect the improvement of his / her utility level by changing his / her schedule. Then, the equilibrium condition can be formulated as follows;

$$\begin{aligned} \dot{W}(q) &= AB^{\alpha}\{(N-q)^{\alpha} - q^{\alpha}\}\dot{n}(q) - (e_1 - c_1)\frac{1}{k} + e_1\dot{a}(q) - e_2\frac{1}{k} + (e_2 - c_2)\dot{l}(q) \\ &= 0 \end{aligned} \tag{10}$$

where, $\dot{}$ denotes a derivative for q.

In order to derive the distributions of equilibrium schedule, we solve the optimal control problem that maximizes social welfare as eq.(9) with the above equilibrium condition (10);

$$\max_{S_1,S_2} SW = \int_0^N W(q)\mathrm{d}q \tag{11}$$

$$\text{s.t.} \quad \dot{W}(q) = 0 \tag{12}$$

For the user equilibrium problem formulated above, we can get the analytic formula of feasible optimal solution. Considering the feasible combinations, we can find that there are only five commuting patterns of optimal solution satisfying the necessary condition (see **Appendix A**).

3.4 System Optimum Model under FWH (First Best)

Let us formulate the system optimum FWH problem. As traffic outflow rate cannot be over k at the bottleneck, no queue situation gives the optimal situation. As a result, $a(q)$ equals to $m(q)$, and $l(q)$ equals to $b(q)$. Then, the system optimum FWH problem can be formulated as the following optimal control problem;

$$\max_{S_1,S_2,s(q)} SW = \int_0^N \left[Y(q,S_1,S_2) - c_1\left\{T_1 - \left(\frac{q}{k} + S_0\right)\right\} - c_2\left\{\frac{q}{k} + (S_1 + H) - T_2\right\}\right]\mathrm{d}q \tag{13}$$

$$\text{s.t.} \quad \dot{Y}(q) = AB^{\alpha}\left\{(N-q)^{\alpha} - q^{\alpha}\right\} s(q) \tag{14}$$

$$\dot{n}(q) \equiv s(q) \tag{15}$$

For the system optimal problem formulated above, we can get the analytic formula of feasible optimal solution. Considering the feasible combinations, we can find that there are only three commuting patterns of optimal solution (see **Appendix B**).

4. NUMERICAL EXAMPLE

In this section, we show numerical examples to use the model formulated in the last section, and illustrate the commuting distribution patterns; home departure / return time, office arrival / leaving time and work start (end) time distributions.

Parameter values for the numerical examples below are given as follows except the temporal agglomeration parameter α; $c_1 = 10$(yen/min), $e_1 = 50$(yen/min), $c_2 = 5$(yen/min), $e_2 = 30$(yen/min), $k = 50$, $N = 5000$, $B = 10$, $H = 450$(min) except break time (=60min), $A = 44.4/(50{,}000)^{\alpha}$. The value of A here certifies $Y = 20000$, when all workers begin to work at a point in time (conventional situation without FWH). When core time starts at 10:00 under FWH, considering the slope of morning schedule cost c_1 is larger than that of evening one c_2, the last worker always begins to work at 10:00. Then, the origin points in time axis are given $T_1 = 10:00$, $T_2 = 16:50$, which is a point the first worker may end work.

4.1 Typical Commuting Patterns in User Equilibrium

(1) Non-flexible pattern

Figure 1 shows the schedule pattern that maximizes the utility level without introducing FWH by firms. The morning part of this pattern is the same situation as analyzed by Arnott R. *et al.*(1990). All workers are obliged to begin to work at 10:00 in the morning and to work until 18:30 in the evening. In this pattern, home departure time and office leaving time distributions are selected to be equal to the disutility levels, respectively. For example, in the morning last commuter who encounters largest congestion wait for 20 minutes to pass the bottleneck, but

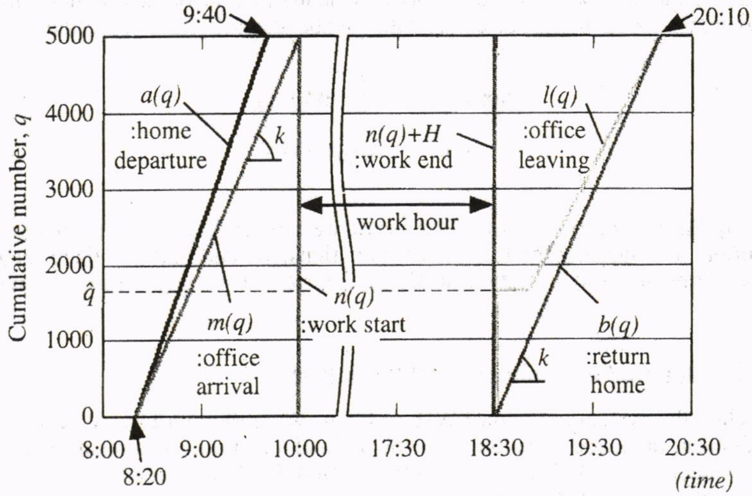

Figure 1. Non-Flexible Pattern under User Equilibrium

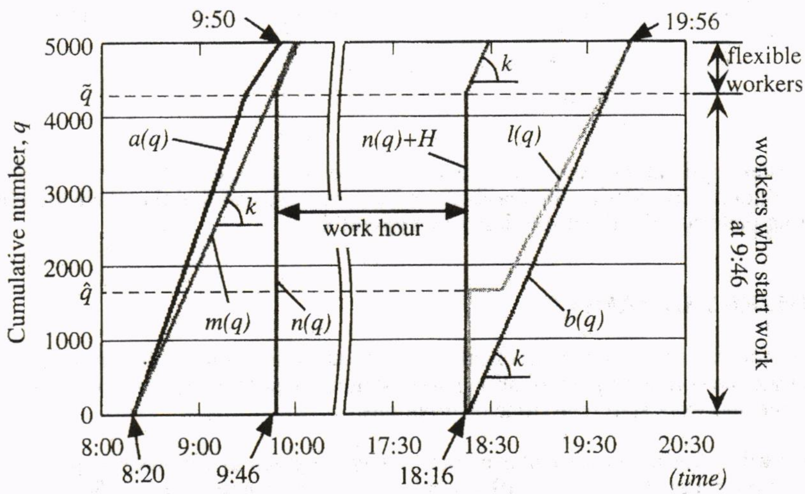

Figure 2. Mixed Pattern 1 under User Equilibrium (α = 0.8)

that cost is compensated by no wait time before work. In the evening, however, such balancing between wait in office and wait in congestion cannot be satisfied in case of the early leavers. If a worker leaves office just after the work end time, he / she may get no waiting and small disutility. Their realized disutility is different according to departure time. Then workers have incentive to try to leave office as early as possible. As a result, many workers make queue at the bottleneck, and the realized departure time of each worker becomes probabilistic. In the equilibrium, the average utility in this queue is the same as others. $1666^{th}(=\hat{q})$ commuter gets smallest utility, and waits for 33 minutes to pass the bottleneck.

As shown afterwards in **Figure 6**, this **non-flexible pattern** cannot be chosen as the optimal pattern in any value of the parameter α, but when α is very large, this pattern may give approximately same value as the optimal pattern.

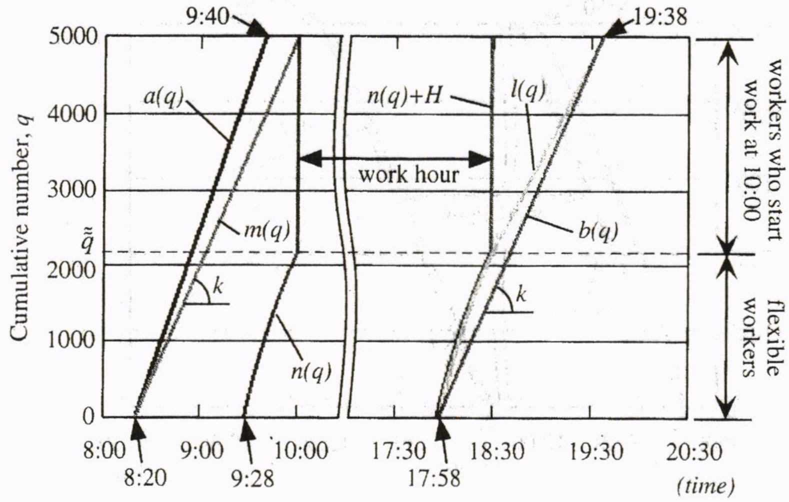

Figure 3. Mixed Pattern 2 under User Equilibrium (α = 0.4)

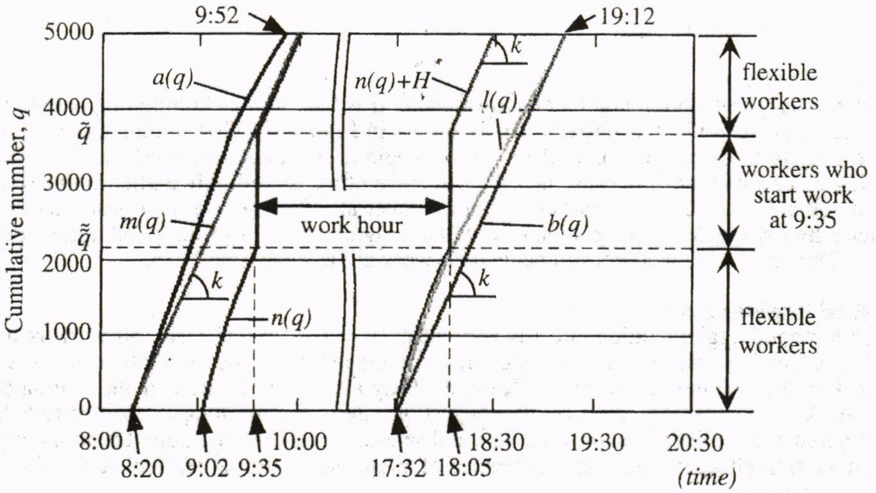

Figure 4. Mixed Pattern 3 under User Equilibrium (α = 0.4)

(2) Mixed pattern 1

Figure 2 shows the commuting pattern that maximizes the utility level when the temporal agglomeration effect is not too strong (α = 0.8, in this case). In this case, about 4300 workers begin to work at same time (9:46), but the other 700 start to work just after their arrival between 9:46 and 10:00 continuously, because agglomeration effect is weaker compared with reduction of schedule cost. In the evening, office leaving time and home return time distributions are the same shapes as the first pattern, but 14 minutes shifted forward because the first worker can end work at 18:16. In this numerical setting, this pattern can be realized by applying FWH system to only about 700 out of 5000 workers even if any firms cannot actually introduce the FWH system.

(3) Mixed pattern 2

Figure 3 shows that earlier workers begin to work earlier than 10:00 continuously. Because it is assumed that c_1 is lager than c_2 in our numerical setting, utility level of this pattern cannot exceed utility level of **pattern (2)**. In other words, this pattern does not become the optimal

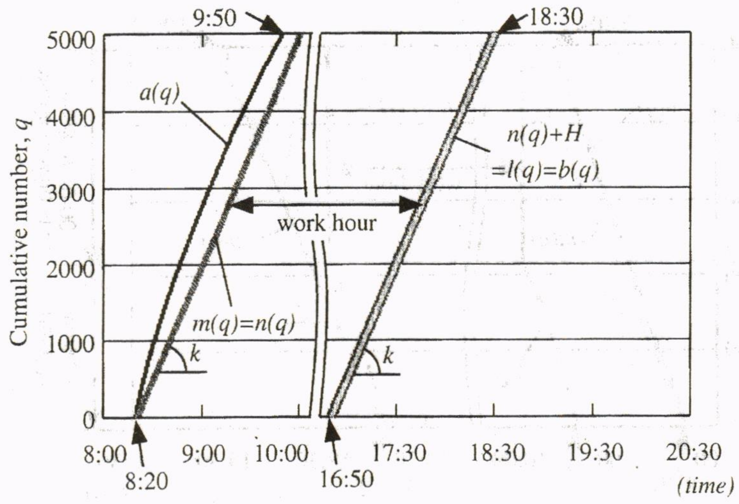

Figure 5. Total Flexible Pattern under User Equilibrium (α = 0.2)

solution in this setting.

(4) Mixed pattern 3
When the temporal agglomeration effect parameter α is 0.4, optimal commuting pattern is the mixture of **pattern (2)** and **pattern (3)**, as shown in **Figure 4**. 2191 commuters who arrive office early to avoid congestion in the morning, begin to work between 9:02 and 9:35 without waiting other commuters, because he / she can leave office accordingly earlier in the evening. On the other hand, 1287 commuters who arrive later at 9:35 begin to work just after arrival, because he / she reduces congestion cost without significant loss of temporal agglomeration effect. The other 1522 workers still begin their work at the same time, 9:35.

(5) Total flexible pattern
When temporal agglomeration effect is very weak (α = 0.2, in this case), all workers utilize FWH system; they start to work just after arrival in the morning, and leave office just after end to work in the evening, as shown in **Figure 5**. They also return home without waiting at the bottleneck in the evening. As a result, congestion rate in the morning is decided to balance the difference of utility based on wage rate and schedule cost. Furthermore, because temporal agglomeration effect is weak, and difference of wage rate is not large, they suffer about the same congestion.

Figure 6 shows the comparison of the utility levels of the above five feasible patterns for several values of the temporal agglomeration effect parameter α. While temporal agglomeration effect is weak ($\alpha < 0.25$), the **total flexible pattern** is best, because the amount of reduction of the schedule cost in the evening and of the congestion cost are larger than the decrease of temporal agglomeration effect. When temporal agglomeration effect becomes strong, the **mixed pattern 1** becomes optimum. Because the **mixed pattern 3** cannot be feasible because of dissatisfying jumping condition, as α becomes smaller 0.39, the **mixed pattern 1** becomes optimum in this values. However, as the parameter α becomes larger ($\alpha > 0.39$), the utility level of the **mixed pattern 3** exceeds the **mixed pattern 1** slightly. When α becomes larger furthermore, the **mixed pattern 1** becomes optimum again ($\alpha > 0.52$). This pattern keeps its superiority over the **non-flexible pattern**.

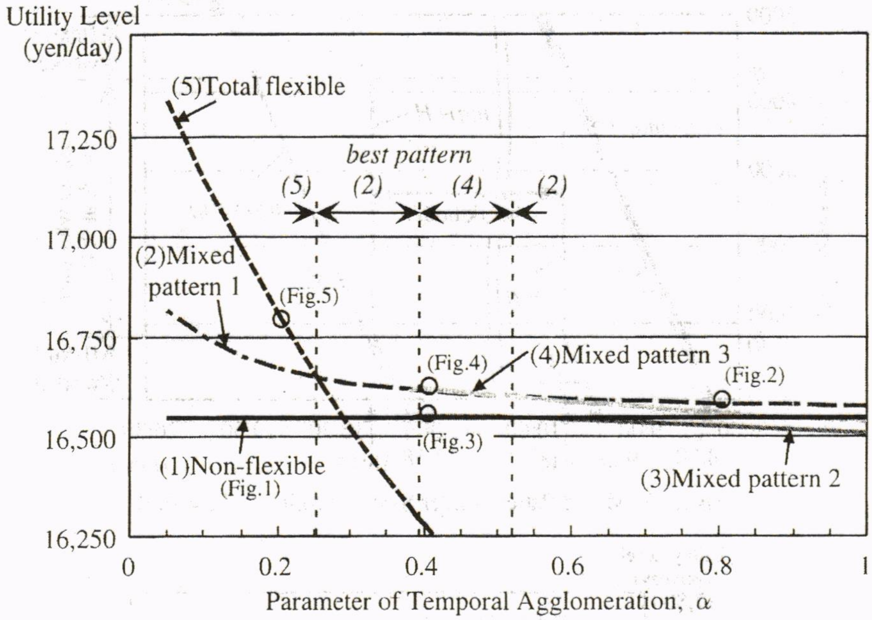

Figure 6. Utility Levels of the Five Patterns under User Equilibrium

4.2 Typical Commuting Patterns in System Optimal Problem

(1) Non-flexible pattern
When temporal agglomeration effect is strong, **non-flexible pattern** becomes optimum. This pattern is a pattern that removes congestion time in both the morning and the evening from **Figure 1**; he / she departs home just before his / her office arrival time, and keeps to wait at his / her office departure just till the return home time, because the slope of congestion cost e_2 is larger than that of schedule cost c_2.

(2) Mixed pattern
Figure 7 shows when α is not large (α = 0.3). 3360 workers start to work at same time, 9:44. Others begin to work before / after 9:44, because total schedule cost can be saved compared with the amount of decreasing agglomeration effect.

(3) Total flexible pattern
When temporal agglomeration effect is very weak, **total flexible pattern** becomes optimum. This pattern is a pattern that removes congestion time in the morning from **Figure 5**; he / she departs home just before his / her work start time (= office arrival time), and leaves his / her office (= return home time) just after his / her work end time.

In this system optimal problem, utility level can be different for each worker. **Figure 8** shows the comparison of the average utility levels of the above three feasible patterns for several values of the temporal agglomeration effect parameter α. While temporal agglomeration effect is weak (α < 0.18), the **total flexible pattern** is best, because the amount of reducing the schedule cost is larger than the loss of agglomeration effect. When temporal agglomeration effect becomes strong, the **mixed pattern** becomes optimum. Furthermore, as the parameter α becomes larger (α > 0.45), the **non-flexible pattern** becomes optimum.

In order to realize the system optimal solution, for example, we must introduce either time dependent peak-load pricing or office tax differentiated by work schedule to compensate the

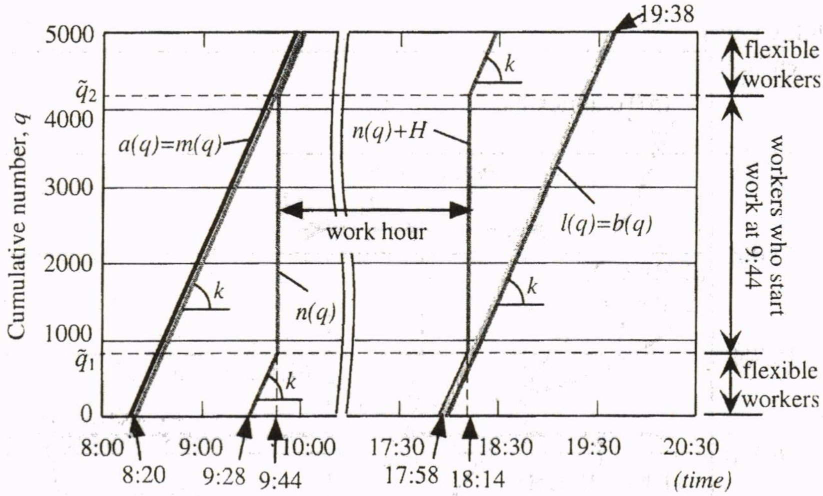

Figure 7. Mixed Pattern under System Optimum (α = 0.3)

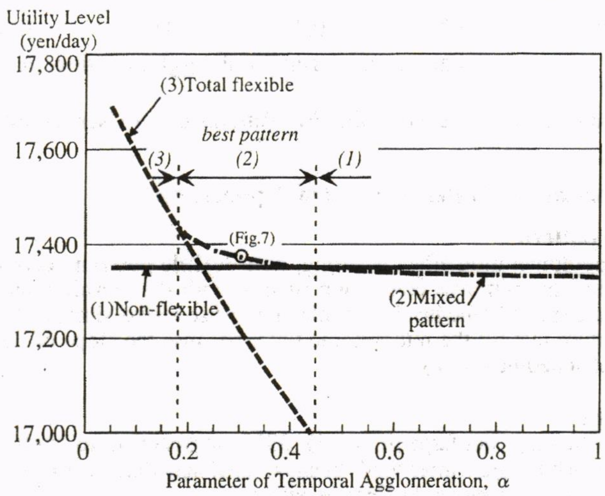

Figure 8. Utility Levels of the Three Patterns under System Optimum

difference of utility level among the workers.

4.3 Comparison with / without FWH under User Equilibrium / System Optimum

(1) Average values of cost items comparison

Figure 9 shows each average value of cost items with / without FWH system when the α is 0.2. We compare (ii)utility level with FWH with (i)utility level of fixed work time under user equilibrium (U.E.), and find effect to introduce the FWH system. In this numerical example, the FWH possessed total effect for 259(yen). In case (ii), wage rate loss is 404(yen), but schedule cost is saved as much as 450(yen). Although congestion cost becomes zero in the evening, we notice that congestion cost increase 87(yen) in the morning by the introduction of the FWH.

Furthermore, we compare the system optimum (S.O.) cases, fixed work time is indexed case

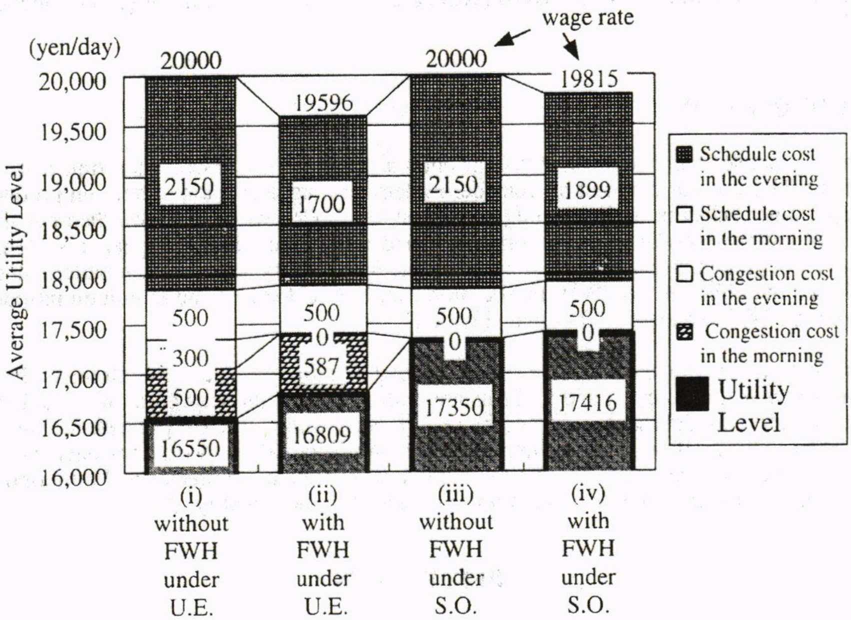

Figure 9. Average Values of Cost Items with / without FWH ($\alpha = 0.2$)

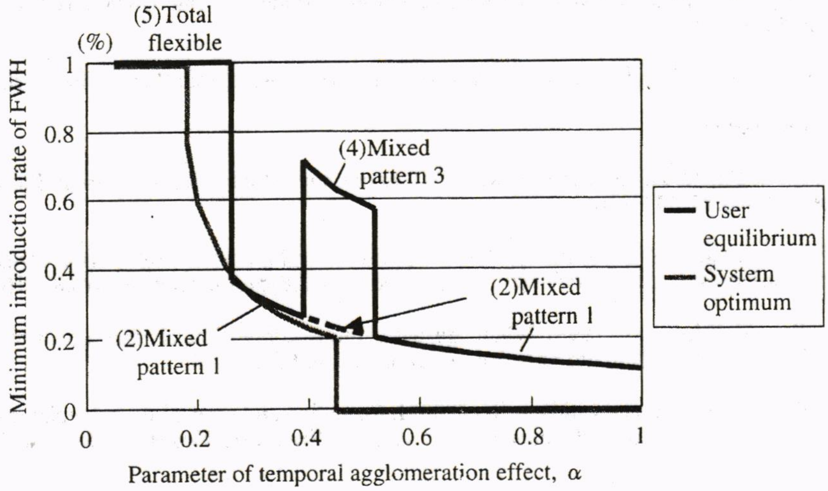

Figure 10. Minimum Introduction Rate of FWH

(iii) and optimal FWH is indexed case (iv). Comparing with the basic case (i), total utilities increase 800(yen), 866(yen), respectively because of no congestion.

(2) FWH rate comparison
To use the above results, we can also calculate the minimum introduction rate of the FWH to realize the optimal solution, as shown in **Figure 10**. If the temporal agglomeration effect parameter α is large than 0.52, the required FWH ratio is not large. Only one fifth of total firms should introduce FWH system to realize the user optimum. When temporal agglomeration effect becomes weak, introduction rate of the FWH increases, but at $0.39 < \alpha < 0.52$ (the **mixed pattern 3**), introduction rate becomes larger. However, the utility level of the **mixed**

pattern 3 is close to that of the **mixed pattern 1**, which do not require high introduction rate of FWH.

5. CONCLUSION

We have proposed the model to analyze the optimal commuting and work start time distribution under FWH on motor commuting considering temporal agglomeration effect on productivity and commuter behavior in the evening. Based on our model, we have shown theoretically that only five patterns appear in user equilibrium, and only three patterns appear in system optimum. Through numerical examples, we have calculated utility level for each pattern, and have confirmed the effect of the FWH. Furthermore, we have calculated the minimum introduction rate of the FWH to realize the optimal situation.

It should be noted, however, that we set very strong assumptions such as commuter's behavior neglects personal difference, and all firms are uniform. From the viewpoint of travel behavior modeling, these models seem far from the reality. But owing to the simplifications, we get analytical tractability. If we combine more reliable parameter values based on behavior analysis, we can expect the market equilibrium and effect of policies more precisely. Furthermore, we need to extend to multi-modal case, in order to approach more reality.

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Appendix A. USER EQUILIBRIUM SOLUTION

For the user equilibrium problem formulated in the **section 3.3**, analytic solutions of the five feasible optimal patterns can be obtained as follows.

(1) Non-flexible pattern
The solutions of the **non-flexible pattern** are represented as follows;

$$
\begin{aligned}
a(q) &= \frac{e_1 - c_1}{e_1 k} q + \left(S_2 - \frac{N}{k}\right) && (16)\\
m(q) &= \frac{1}{k} q + \left(S_2 - \frac{N}{k}\right) && (17)\\
n(q) &= S_1 = S_2 && (18)\\
l(q) &= \frac{e_2}{(e_2 - c_2)\,k}(q - N) + \left(S_1 + H + \frac{N}{k}\right) && (19)\\
b(q) &= \frac{1}{k} q + (S_1 + H) && (20)
\end{aligned}
$$

Our model assumes that there is no traffic flow at the bottleneck except commuting time. Under the **non-flexible pattern**, if commuter can arrive first at the bottleneck, he / she gets smallest disutility, because he / she passes without congestion and return home earliest. Then some people who want to get smaller disutility try to leave office just after work end time, but if he / she cannot line up earlier at the bottleneck, he / she will suffer disutility larger than average disutility of evening part. In order to represent this phenomenon, we define office leaving time function $l(q)$ as follows again;

$$
l(q) = \begin{cases} S_1 + H & \text{if} \quad q \le \dfrac{2c_2 N}{e_2} \\[2ex] \dfrac{e_2}{(e_2 - c_2)\,k}(q - N) + \left(S_1 + H + \dfrac{N}{k}\right) & \text{if} \quad q > \dfrac{2c_2 N}{e_2} \end{cases} \tag{21}
$$

The above function means disutility of $(c_2 N/e_2)^{th} (\equiv \hat{q}/2)$ commuter who are middle number of commuters to leave office at $S_1 + H$ equals to the average disutility.

(2) Mixed pattern 1
This pattern shows that earlier commuters are non-flexible work and later ones take flexible work pattern. The solutions for $m(q)$, $l(q)$ and $b(q)$ are the same as **pattern (1)**. Other solutions, $a(q)$ and $n(q)$ are divided at $\tilde{q}$ which means the number of the non-flexible workers, can be given with the following formula;

$$
a(q) = \begin{cases} \dfrac{e_1 - c_1}{e_1 k} q + \left(S_2 - \dfrac{N}{k}\right) & \text{if} \quad q \le \tilde{q} \\[2ex] \dfrac{AB^{\alpha}}{e_1 k(\alpha + 1)}\left[\left\{(N - q)^{\alpha+1} + q^{\alpha+1}\right\} - \left\{(N - \tilde{q})^{\alpha+1} + \tilde{q}^{\alpha+1}\right\}\right] & \\[2ex] \quad + \dfrac{e_1 - c_1}{e_1 k} q + \left(S_2 - \dfrac{N}{k}\right) & \text{if} \quad q > \tilde{q} \end{cases} \tag{22}
$$

$$n(q) = \begin{cases} S_1 & \text{if} \quad q \leq \tilde{q} \\ \frac{1}{k}q + \left(S_2 - \frac{N}{k}\right) \quad (= m(q)) & \text{if} \quad q > \tilde{q} \end{cases} \tag{23}$$

(3) Mixed pattern 2

In this pattern, earlier commuters take flexible work and later ones are non-flexible workers. The solutions for $a(q)$, $m(q)$ and $b(q)$ are the same as **pattern (1)**. On the other hand, $n(q)$ and $l(q)$ are divided at $\tilde{\tilde{q}}$ indicating the proportion of flexible workers,

$$\begin{cases} \dot{l}(q) = \left(\frac{e_2}{k}\right)\left(\frac{1}{AB^\alpha\{(N-q)^\alpha - q^\alpha\} + e_2 - c_2}\right) & \\ \qquad l(0) = S_1 + H,\ l(\tilde{\tilde{q}}^-) = l(\tilde{\tilde{q}}^+) & \text{if} \quad q < \tilde{\tilde{q}} \\ l(q) = \frac{e_2}{(e_2 - c_2)k}(q - N) + \left(S_1 + H + \frac{N}{k}\right) & \text{if} \quad q \geq \tilde{\tilde{q}} \end{cases} \tag{24}$$

$$n(q) + H = \begin{cases} l(q) & \text{if} \quad q < \tilde{\tilde{q}} \\ S_2 + H & \text{if} \quad q \geq \tilde{\tilde{q}} \end{cases} \tag{25}$$

where, the first equation of eq.(24) is differential equation for q with initial condition, as well as terminal condition, that both functions for eq.(24) take the same value at $\tilde{\tilde{q}}$.

(4) Mixed pattern 3

This pattern contains three types of workers; middle commuters are non-flexible work, and at both sides workers take flexible work. That is represented as the combination of **pattern (2)** and **pattern (3)**, and described by eq.(17), (20) and (22)-(25). Two switching points $\tilde{q}$ and $\tilde{\tilde{q}}$ can be determined by initial and terminal conditions.

(5) Total flexible pattern

The solutions of the **total flexible pattern** are represented as follows;

$$a(q) = \frac{e_1 - c_1 + c_2}{e_1 k}q + \frac{AB^\alpha}{e_1 k(\alpha+1)}\left\{(N-q)^{\alpha+1} + q^{\alpha+1} - N^{\alpha+1}\right\} + \left(S_2 - \frac{N}{k}\right) \tag{26}$$

$$m(q) = n(q) = \frac{1}{k}q + \left(S_2 - \frac{N}{k}\right) \tag{27}$$

$$n(q) + H = l(q) = b(q) = \frac{1}{k}q + (S_1 + H) \tag{28}$$

Appendix B. SYSTEM OPTIMAL SOLUTION

For the system optimal problem formulated in the **section 3.4**, the analytic solutions of the three feasible optimal solution patterns can be obtained as follows.

(1) Non-flexible pattern

The solutions of the **non-flexible pattern** are represented as follows;

$$a(q) = m(q) = \frac{1}{k}q + \left(S_2 - \frac{N}{k}\right) \tag{29}$$

$$n(q) = S_1 = S_2 \tag{30}$$

$$l(q) = b(q) = \frac{1}{k}q + (S_1 + H) \tag{31}$$

(2) Mixed pattern
In this pattern, middle commuters take non-flexible work, and the others at both sides are flexible workers. The solutions of $a(q)$, $m(q)$, $l(q)$ and $b(q)$ are the same as **pattern (1)**.

$$n(q) = \begin{cases} l(q) - H & \text{if} \quad q < \tilde{q}_1 \\ S_1 + \dfrac{\tilde{q}_1}{k} & \text{if} \quad \tilde{q}_1 \le q \le \tilde{q}_2 \\ m(q) & \text{if} \quad q > \tilde{q}_2 \end{cases} \tag{32}$$

where, $\tilde{q}_1$, $\tilde{q}_2\,(> \tilde{q}_1)$: switching point.

(3) Total flexible pattern
The solutions of the **total flexible pattern** are represented as follows;

$$a(q) = m(q) = n(q) = \frac{1}{k}q + \left(S_2 - \frac{N}{k}\right) \tag{33}$$

$$n(q) + H = l(q) = b(q) = \frac{1}{k}q + (S_1 + H) \tag{34}$$