A CONTINUOUS TRAFFIC EQUILIBRIUM MODEL WITH MULTIPLE USER CLASSES

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Abstract: We consider a city with several facilities that compete with each other for customers of different classes. The transportation system of the city is a highly dense road network. The customers are distributed continuously in the city, and each chooses a facility based on his or her transportation cost to the facility and the in-situ cost at the facility. We assume that different classes of users perceive costs differently, and hence are subject to different transportation cost functions and market externalities. An equivalent mathematical model is formulated and proved to satisfy the user equilibrium conditions. The resultant problem is solved by the finite element method. Numerical examples for both cases of fixed and elastic demand are given to illustrate the effectiveness of the proposed methodology.

Key Words: continuous model, traffic equilibrium, multiple user classes, finite element method, Newtonian algorithm

1. INTRODUCTION

For the equilibrium prediction and solution algorithm of the flow in a transportation system, there are, in general, two approaches in the literature: discrete modeling and continuous modeling. In the discrete approach, different zones are treated as centroids, and roads are considered as nodes and links that join these centroids to form a discrete network. In the continuous approach, the major assumption is that the variation in the nearby areas is relatively small when compared to the entire system. Thus, the characteristics of the transportation system, such as flow and cost, can be represented by smooth mathematical functions (Vaughan, 1987).

Traditionally, traffic equilibrium problems are studied with the discrete modeling approach. However, much attention has recently been paid to the continuous modeling approach to cope with the deficiencies of the discrete approach. First, for the strategic planning of a very dense transportation system, the discrete approach involves a large number of unknowns that are generally difficult to solve in an efficient manner. Second, it has to assume that the demand is concentrated at a centroid, which is unrealistic because the demand is distributed over the region. Thus, the continuous approach can model these aspects in a more effective manner.

A sizeable literature focuses on the continuous modeling approach. In some of the early studies, such as those of Buckley (1979), Dafermos (1980), D'Este (1987), and Wong (1994), the continuous models for cities were formulated with specific shapes (such as circular or rectangular), which was not adequate to model real life situations. Some other studies considered an arbitrary shape for the city configuration (Beckmann, 1952; Zitron, 1974; Williams and Ortuzar, 1976). Sasaki *et al.* (1990) further extended the idea of continuous modeling for a city with arbitrary shape and used the finite element method to solve the resultant problems. Wong *et al.* (1998) developed a more robust finite element procedure to solve the problem. Based on this framework, a number of extensions were made to apply the continuous modeling approach to traffic equilibrium problems, which include the estimation of market areas (Wong and Yang, 1999; Yang and Wong, 2000), and simultaneous discrete and continuous modeling (Yang *et al*., 1994; Wong *et al*., 2002a). For practical applications, Ho *et al*. (2002) has successfully applied the continuum model for the airport coordination problem in the Pearl River Delta region in China.

All of the above continuum models for arbitrary city shapes were for single user classes, which is unrealistic because, in practical terms, networks are shared by different types of users whose perceived travel cost and congestion externality can be very different. A single user class cannot closely model the real situation. Hence, it is necessary to consider multiple user classes in the continuous modeling method (Wong, 1998; Wong and Sun, 2001; Wong *et al*., 2002b). The present research is inspired by that of Vliet *et al*. (1986), which was developed for the discrete network that models how different classes of users interact with each other.

We develop a continuous equilibrium model with multiple user classes, in which each class of users has a distinct cost function. The cases of elastic demand and market externality are also considered. In Section 2, the modeled city with competitive facilities, together with the customer's behavioral assumptions, are introduced. The formulation of a multi-class user equilibrium problem as an equivalent mathematical program is given in Section 3. Section 4 derives a finite element solution algorithm to solve the resultant problem. Section 5 presents two numerical examples to illustrate the effectiveness of the proposed methodology.

2. THE MODELLED CITY

Consider a city in a two-dimensional plane as illustrated in Figure 1, in which the road network is approximated as a continuum. In this city, there are N facilities (such as shopping centers, industrial centers, or public transportation terminals) competing with each other to attract customers of M different classes whose home locations are spread over the whole city. The dense transportation network is approximated as a continuum. The detailed treatment of the relationship and conversion between the dense discrete network and the approximated continuum can be found in Sasaki *et al*. (1990). The customers travel from their home locations through the continuum to patronise any one of the facilities. Denote the city as Ω, the boundary of the study region as Γ, and the location of the nth facility as O_n.

The distribution of demand for class m customers over the city, Ω, is assumed to be continuous and represented by a non-negative, heterogeneous density function $q_m(x,y)$, where $q_m(x,y)$ is the total demand per unit area, from location $(x,y)\in\Omega$, to patronize any one of the facilities. For the case of elastic demand, the traffic demand of a particular class of customers at any point within the city is directly related to their total transportation cost from that point to their chosen facility plus the in-situ cost at the facility. The demand function can be specified as

$$q_m(x,y) = D_m(x,y,u_m(x,y)) \tag{1}$$

where $u_m(x,y)$ is the total transportation and in-situ cost for class m customers from the point $(x,y)\in\Omega$ to the chosen facility, and the demand function $D_m(.)$ is assumed to be a monotonic decreasing function with respect to the total cost at that point. Denote $\mathbf{q} = (q_m(x,y), m = 1,2,\ldots,M)$. The transportation cost for each class of customers in the region is assumed to be dependent on the total local flow intensity and road configuration, but not upon direction of any class of users (an isotopic case),

$$c_m(x,y,\mathbf{f}) = a_m(x,y) + \sum_{l=1}^{M} b_{ml}(x,y)|\mathbf{f}_l(x,y)| \tag{2}$$

where $\mathbf{f} = (\mathbf{f}_m(x,y), m = 1,2,\ldots,M)$, $c_m(x,y,\mathbf{f})$ is the transportation cost of class m customers to travel a unit distance at co-ordinate $(x,y)\in\Omega$, $a_m(x,y)$ and $b_{ml}(x,y)$ are strictly positive scalar functions, $\mathbf{f}_m(x,y) = (f_{mx}(x,y), f_{my}(x,y))$ is a vector that represents the flow state of customer class m in the city, $f_{mx}(x,y)$ and $f_{my}(x,y)$ are respectively the flow flux in the x and y directions, and

$$|\mathbf{f}_m(x,y)| = \sqrt{f_{mx}(x,y)^2 + f_{my}(x,y)^2} \tag{3}$$

is the norm of the flow vector for class m customers at (x,y). For the cost function, the first term represents the free flow (zero-flow) transportation cost per unit distance, and the second term reflects the effects of congestion. The explicit dependence of the cost function $c_m(x,y,\mathbf{f})$ on location (x,y) reflects the possibility that the density and capacity of the transportation system may vary from point to point. Here, we consider a symmetric case where $b_{ml} = b_{lm}$, $\forall m,l$.

Inside the domain of the city Ω, the flow vector and customers' demand must satisfy the flow conservation condition, and this can be specified by

$$\nabla \mathbf{f}_m + q_m = 0, \quad \forall (x,y) \in \Omega, \forall m = 1,\ldots,M \tag{4}$$

Assuming that no flow crosses the boundary of the city, we have

$$\mathbf{f}_m = 0, \quad \forall (x,y) \in \Gamma, \forall m = 1,\ldots,M \tag{5}$$

Note that it is straightforward to extend the model to deal with the case of $\mathbf{f}_m \cdot \mathbf{n} = g_m(x,y)$ on the boundary Γ, where $\mathbf{n}$ is the normal vector on the boundary and g_m is a function that represents the given demand distribution of class m customers entering or leaving the city through the boundary. To avoid singularity at the facilities, we assume that each of the facilities are of a finite size enclosed by a clockwise boundary segment Γ_{cn}, n = 1, 2,…,N.

Denote Ω_{mn} as the market area of facility n for class m customers, and the number of class m customers that will be attracted to facility n becomes

$$Q_{mn} = \iint_{\Omega_{mn}} q_m(x,y)\mathrm{d}\Omega, \quad m = 1,2,\ldots,M, \; n = 1,2,\ldots,N \tag{6}$$

From the flow conservation principle at the peripheral of facility n we have

$$\int_{\Gamma_{cn}} \mathbf{f}_m \cdot \mathbf{n} \, \mathrm{d}\Gamma - Q_{mn} = 0, \quad m = 1,2,\ldots,M, \; n = 1,2,\ldots,N \tag{7}$$

Within the city with more than one facility, customers' choice of facility is not only based on their transportation cost to each of the facility, but also on the in-situ costs at the facilities. The definition of in-situ cost varies for different types of facilities, such as the average prices of consumers' products in shopping centers, parking costs at airports, etc. Customers make their choices of facility as well as the routes taken to maximize their utilities. Therefore, we define a generalized cost function to represent the total cost perceived by a customer of class m to patronize a facility as a function of the relevant attributes of the in-situ costs at the facility and transportation cost to that facility,

$$G_m[H(x,y),O_n] = \sum_{k=1}^{K} w_{mk} p_{mnk} + C_m[H(x,y),O_n] = \bar{u}_{mn} + C_m[H(x,y),O_n] \tag{8}$$

where $G_m[H(x,y),O_n]$ is the generalized cost for class m customers from their home location $H(x,y)$ to patronize facility n (n = 1, 2, …, N). $\bar{u}_{mn} = \sum_{k=1}^{K} w_{mk} p_{mnk}$ represents the total in-situ cost at the facility n, where K attribute measures p_{mnk} for k =1,...,K are considered for the facility n, each with an associated weight w_{mk} perceived by class m customers. $C_m[H(x,y),O_n]$ is the minimum transportation cost for class m customers to travel from home location $H(x,y)$ to their chosen facility n. In our model, the weights and the attribute measures are exogenous inputs and their values are given, whereas the equilibrium transportation cost are determined from user-equilibrium conditions.

As discussed previously, the customers patronize the facility with the lowest generalized cost in a deterministic manner. The individual class m customer from home location $H(x,y)$ will choose to patronize facility n, so that the following equation is satisfied:

$$G_m[H(x,y),O_n] \geq G_m[H(x,y),O_l] \quad \forall l = 1,2,\ldots,N \tag{9}$$

Furthermore, with the chosen facility, customers will make their routing decision over the continuum city space in a user-optimal manner, so that the total path cost is minimized:

$$C_m[H(x,y),O_n] = \min_p C_{mp}[H(x,y),O_n] \quad \forall p \tag{10}$$

where $C_{mp}[H(x,y),O_n] = \int_p c_m(x,y,\mathbf{F})\mathrm{d}s$ is the transportation cost from $H(x,y)$ to O_n, via path p in the continuum space.

3. PROBLEM FORMULATION

The problem of a multi-class customer spatial choice equilibrium with elastic demand can be formulated as the following minimization problem.

$$\underset{\mathbf{f},\mathbf{q}}{\text{Minimise}} \quad z = \sum_{m=1}^{M}\sum_{n=1}^{N}\bar{u}_{mn}Q_{mn} + \iint_{\Omega}\left(\sum_{m=1}^{M}a_m|\mathbf{f}_m| + \sum_{m=1}^{M}\sum_{l=1}^{M}\frac{b_{mn}}{2}|\mathbf{f}_m||\mathbf{f}_l| - \sum_{m=1}^{M}\int_0^{q_m}D_m^{-1}(\xi)\mathrm{d}\xi\right)\mathrm{d}\Omega \tag{11a}$$

subject to

$$\nabla\mathbf{f}_m + q_m = 0\,, \forall(x,y)\in\Omega,\ \forall m = 1,\ldots,M \tag{11b}$$

$$\mathbf{f}_m = 0\,, \forall(x,y)\in\Gamma\,,\ \forall m = 1,\ldots,M \tag{11c}$$

$$Q_{mn} + \int_{\Gamma_{cn}}\mathbf{f}_m\cdot\mathbf{n}\,\mathrm{d}\Gamma = 0\,,\ \forall(x,y)\in\Gamma_{cn}\,,\ \forall n = 1,\ldots,N, \forall m = 1,\ldots,M \tag{11d}$$

Consider the following Lagrangian function:

$$\Pi = \iint_{\Omega}\sum_{m=1}^{M}a_m|\mathbf{f}_m| + \sum_{m=1}^{M}\sum_{l=1}^{M}\frac{b_{ml}}{2}|\mathbf{f}_m||\mathbf{f}_l| - \sum_{m=1}^{M}\int_0^{q_m}D_m^{-1}(\xi)\mathrm{d}\xi + \sum_{m=1}^{M}u_m(\nabla\mathbf{f}_m + q_m)\mathrm{d}\Omega$$
$$\int_{\Gamma}\sum_{m=1}^{M}\mathbf{w}_m\cdot\mathbf{f}_m\,\mathrm{d}\Gamma + \sum_{m=1}^{M}\sum_{n=1}^{N}\bar{u}_{mn}Q_{mn} + \sum_{m=1}^{M}\sum_{n=1}^{N}\pi_{mn}\left(Q_{mn} + \int_{\Gamma_{cn}}\mathbf{f}_m\cdot\mathbf{n}\,\mathrm{d}\Gamma\right) \tag{12}$$

where $u_m, \mathbf{w}_m = (w_{mx}, w_{my})$, and π_{mn} are the Lagrangian multipliers associated with equations (11b), (11c), and (11d), respectively. Applying variational principle to the Lagrangian function, we can show that

$$\delta\Pi = \iint_{\Omega}\sum_{m=1}^{M}\frac{a_m\mathbf{f}_m\cdot\delta\mathbf{f}_m}{|\mathbf{f}_m|} + \sum_{m=1}^{M}\sum_{l=1}^{M}\frac{b_{ml}}{2}\left(\frac{\mathbf{f}_m\cdot\delta\mathbf{f}_m|\mathbf{f}_l|}{|\mathbf{f}_m|} + \frac{\mathbf{f}_l\cdot\delta\mathbf{f}_l|\mathbf{f}_m|}{|\mathbf{f}_l|}\right)$$
$$-\sum_{m=1}^{M}D_m^{-1}(q_m)\delta q_m + \sum_{m=1}^{M}\delta u_m(\nabla\mathbf{f}_m + q_m) + \sum_{m=1}^{M}(u_m\nabla\delta\mathbf{f}_m + u_m\delta q_m)\mathrm{d}\Omega$$
$$+\int_{\Gamma}\sum_{m=1}^{M}(\mathbf{w}_m\cdot\delta\mathbf{f}_m + \mathbf{f}_m\cdot\delta\mathbf{w}_m)\mathrm{d}\Gamma + \sum_{m=1}^{M}\sum_{n=1}^{N}\bar{u}_{mn}\delta Q_{mn} + \sum_{m=1}^{M}\sum_{n=1}^{N}Q_{mn}\delta\bar{u}_{mn}$$
$$+\sum_{m=1}^{M}\sum_{n=1}^{N}\left(Q_{mn} + \int_{\Gamma_{cn}}\mathbf{f}_m\cdot\mathbf{n}\,\mathrm{d}\Gamma\right)\delta\pi_{mn} + \sum_{m=1}^{M}\sum_{n=1}^{N}\pi_{mn}\delta Q_{mn} - \sum_{m=1}^{M}\sum_{n=1}^{N}\int_{\Gamma_{cn}}\pi_{mn}\mathbf{n}\cdot\delta\mathbf{f}_m\,\mathrm{d}\Gamma \tag{13}$$

As $\nabla(u_m\delta\mathbf{f}_m) = \delta\mathbf{f}_m\nabla u_m + u_m\nabla\delta\mathbf{f}_m, \forall m = 1,\ldots,M$, and by the divergence theorem, we have

$$\sum_{m=1}^{M}\iint_{\Omega}\nabla(u_m\delta\mathbf{f}_m)\mathrm{d}\Omega = \sum_{m=1}^{M}\int_{\Gamma}u_m\delta\mathbf{f}_m\cdot\mathbf{n}\,\mathrm{d}\Gamma + \sum_{m=1}^{M}\sum_{n=1}^{N}\int_{\Gamma_{cn}}u_m\delta\mathbf{f}_m\cdot\mathbf{n}\,\mathrm{d}\Gamma \tag{14}$$

Substituting equation (14) into equation (13), we can show that

$$\delta\Pi = \sum_{m=1}^{M}\iint_{\Omega}\left[\left(\left(a_m + \sum_n b_{mn}|\mathbf{f}_n|\right)\frac{\mathbf{f}_m}{|\mathbf{f}_m|} - \nabla u_m\right]\cdot \delta\mathbf{f}_m + \left[u_m - D_m^{-1}(q_m)\right]\delta q_m + (\nabla\mathbf{f}_m + q_m)\delta u_m \,\mathrm{d}\Omega$$
$$+ \sum_{m=1}^{M}\int_{\Gamma}(\mathbf{w}_m + u_m\mathbf{n})\cdot\delta\mathbf{f}_m + \mathbf{f}_m\cdot\delta\mathbf{w}_m \,\mathrm{d}\Gamma + \sum_{m=1}^{M}\sum_{n=1}^{N}\int_{\Gamma_{cn}}(u_m + \pi_{mn})\mathbf{n}\cdot\delta\mathbf{f}_m \,\mathrm{d}\Gamma$$
$$+ \sum_{m=1}^{M}\sum_{n=1}^{N}(\bar{u}_{mn} + \pi_{mn})\delta Q_{mn} + \sum_{m=1}^{M}\sum_{n=1}^{N} Q_{mn}\delta\bar{u}_{mn} + \sum_{m=1}^{M}\sum_{n=1}^{N}\left(Q_{mn} + \int_{\Gamma_{cn}}\mathbf{f}_m\cdot\mathbf{n}\,\mathrm{d}\Gamma\right)\delta\pi_{mn} \quad (15)$$

Because $\delta\mathbf{f}_m$ vanishes on Γ and $\delta\bar{u}_{mn}$ vanishes on Γ_{cn}, $\delta\mathbf{f}_m$, δq_m, and δu_m are arbitrary functions in Ω, $\delta\mathbf{w}_m$ is an arbitrary function on Γ, and $\delta\mathbf{f}_m$, δQ_{mn}, and $\delta\pi_{mn}$ are arbitrary functions on Γ_{cn}, we can easily show that for the stationary point of the Lagrangian $\delta\Pi = 0$,

$$\left(a_m + \sum_n b_{mn}|\mathbf{f}_n|\right)\frac{\mathbf{f}_m}{|\mathbf{f}_m|} - \nabla u_m = 0, \forall(x,y)\in\Omega, m = 1,\ldots,M \quad (16)$$

$$u_m - D_m^{-1}(q_m) = 0, \forall(x,y)\in\Omega, m = 1,\ldots,M \quad (17)$$

$$\nabla\mathbf{f}_m + q_m = 0, \forall(x,y)\in\Omega, m = 1,\ldots,M \quad (18)$$

$$\mathbf{f}_m = 0, \forall(x,y)\in\Gamma, m = 1,\ldots,M \quad (19)$$

$$Q_{mn} + \int_{\Gamma_{cn}}\mathbf{f}_m\cdot\mathbf{n}\,\mathrm{d}\Gamma = 0, \forall(x,y)\in\Gamma_{cn}, n = 1,\ldots,N, m = 1,\ldots,M \quad (20)$$

$$\bar{u}_{mn} + \pi_{mn} = 0, \forall(x,y)\in\Omega, n = 1,\ldots,N, m = 1,\ldots,M \quad (21)$$

$$u_m + \pi_{mn} = 0, \forall(x,y)\in\Gamma_{cn}, n = 1,\ldots,N, m = 1,\ldots,M \quad (22)$$

From (16), we have $\mathbf{f}_m // \nabla u_m$, where "//" means that the vectors are parallel, and

$$|\nabla u_m| = a_m + \sum_{l=1}^{M} b_{ml}|\mathbf{f}_l| = c_m \quad (23)$$

Therefore, $|\nabla u_m|$ can be interpreted as the transportation cost per unit distance for class *m* customers. From (21) and (22), we have

$$u_m = \bar{u}_{mn}, \forall(x,y)\in\Gamma_{cn}, n = 1,\ldots,N, m = 1,\ldots,M \quad (24)$$

This implies that the cost potential u_m at the periphery of facility *n* has the same value as the in-situ cost as perceived by class *m* customers at that facility. For any used path *p* of class *m* customers from home location (H) to facility *n*, the generalized cost is

$$G_p = \bar{u}_{mn} + \int_p c_m \,\mathrm{ds} = \bar{u}_{mc} + \int_p |\nabla u_m|\,\mathrm{ds} = \bar{u}_{mn} + \int_p \nabla u_m\cdot\mathbf{ds} = u_m(H) \quad (25)$$

For any unused path $\tilde{p}$ of class *m* customers from H to facility *n*, the generalized cost is

$$G_{\tilde{p}} = \bar{u}_{mn} + \int_{\tilde{p}} c_m \,\mathrm{ds} = \bar{u}_{mn} + \int_{\tilde{p}} |\nabla u_m|\,\mathrm{ds} \geq \bar{u}_{mn} + \int_{\tilde{p}} \nabla u_m\cdot\mathbf{ds} = u_m(H) \quad (26)$$

From the above two equations, the generalized costs of all the used paths for class m customers are equal, and are less than or equal to those of the unused paths. In the city with more than one facility, we can interpret this as customers choosing the facility for which the total cost for them to travel and use is the minimum. Moreover, for the same chosen facility, customers will choose the path of least transportation cost. Thus, this model ensures that the customers of any class, who have perfect information about the transportation system, make rational decisions in a user-optimal manner (Wardrop, 1952).

4. SOLUTION ALGORITHM

We use the finite element method (Zienkiewicz and Taylor, 1989) to solve the minimization problem (11). The whole city is first discretized into a set of triangular finite elements (Cheung *et al.*, 1996), as shown in Figure 2. Although the continuum city region is divided into smaller sub-regions via the discretization process during the solution stage, the continuum method is still superior than the discrete network approach at least at a global level of modeling, as the size of each element is flexible and each element may already represent thousands of street sections. Moreover, as the traffic demand generated from the housing or other developments is largely continuously distributed within the city, the local variation of demand on a two-dimensional space can be more realistically modeled by piece-wise continuous functions within each element, and the global variation of demand can then be represented by a number of individual elements joining up to form the city region. This contrasts with the discrete modeling approach where the continuous traffic demand has to be modeled by assuming that the demand in a zone is concentrated at a point called centroid and they are assigned to the network through some pre-specified centroid connectors.

The Lagrangian of the minimization problem is now modified as follows:

$$\Pi = \sum_{e\in\Omega}\sum_{m=1}^{M}\iint_{\Omega_e} a_m|\mathbf{f}_m| + \sum_{l=1}^{M}\frac{b_{ml}}{2}|\mathbf{f}_m||\mathbf{f}_l| - \int_0^{q_m} D_m^{-1}(\xi)\mathrm{d}\xi + u_m(\nabla\mathbf{f}_m + q_m)\mathrm{d}\Omega$$
$$\sum_{e\in\Omega}\sum_{m=1}^{M}\int_{\partial\Omega\cap\Gamma}\mathbf{w}_m\cdot\mathbf{f}_m\mathrm{d}\Gamma - \sum_{n=1}^{N}\sum_{e\in\Omega}\sum_{m=1}^{M}\int_{\partial\Omega\cap\Gamma_{cn}}\bar{u}_{mn}(\mathbf{f}_m\cdot\mathbf{n})\mathrm{d}\Gamma$$
$$+\sum_{n=1}^{N}\sum_{e\in\Omega}\sum_{m=1}^{M}\int_{\partial\Omega\cap\Gamma_{cn}}\sigma(u_m - \bar{u}_{mn})\mathrm{d}\Gamma \qquad (27)$$

where Ω_e is the sub-domain of an element e, a_m and b_{ml}, $l = 1, \ldots, M$, are the coefficients of the cost-flow relationship for class m customers that are assumed to be constant within the element. $\mathbf{f}_m$ and u_m are respectively the flow vector and cost potential for the class m customers. $\partial\Omega\cap\Gamma_{cn}$ and $\partial\Omega\cap\Gamma$ indicate the elements that contain, respectively, the boundary of the facilities and the outer bound of the modeled city. $\mathbf{w}$ and σ are, respectively, the Lagrange multipliers along Γ and Γ_{cn}. The solution of the minimization problem in equation (11) can be found by locating the stationary point of the Lagrangian in equation (27). For the second term of equation (27), because there is no traffic flow along the boundary Γ it can be replaced by a much stronger and convenient expression,

$$\sum_{i=1}^{N_O}\sum_{m=1}^{M} w_{mxi} f_{mxi} + w_{myi} f_{myi} \,, \tag{28}$$

where N_O is the number of nodes along Γ, f_{mxi} and f_{myi} are, respectively, the flow flux in the x and y directions at node i for class m, and w_{mxi} and w_{myi} are the corresponding Lagrange multipliers. The zero boundary flow condition can be guaranteed by forcing $f_{mxi} = f_{myi} = 0$ for all nodes along Γ. As the third term of that equation (27) represents the sum of the total customer costs of using each of the facilities, it can be replaced by a much more convenient expression,

$$-\sum_{n=1}^{N}\sum_{i=1}^{S_n}\sum_{m=1}^{M} \bar{u}_{mn} L_i \left(\frac{\mathbf{f}_{mj} + \mathbf{f}_{mk}}{2}\right) \cdot \mathbf{n}_i \tag{29}$$

where S_n is the number of line segments on the boundary of facility n, L_i , and $\mathbf{n}_i$ are, respectively, the length and the unit normal vector of the line segment i around the boundary of the facility n, and $\mathbf{f}_{mj}$, and $\mathbf{f}_{mk}$ are, respectively, the flow vector of the initial and end nodes of the line segment i. Similarly, as it is proved that the cost potential at the boundary of the facility is equal to the total attribute measure of that facility, the last term of equation (27) can be replaced by the following expression,

$$\sum_{n=1}^{N}\sum_{i=1}^{N_n}\sum_{m=1}^{M} \sigma_i \left(u_{mi} - \bar{u}_{mn}\right) \tag{30}$$

where N_n is the number of nodes on the boundary of facility n, u_{mi} is the cost potential of class m at node i, and σ_i is its Lagrange multiplier at node i. The conditions of equation (30) can be ensured by forcing $u_{mi} - \bar{u}_{mc} = 0$ for all nodes along the boundaries of facilities. With all of the above modifications, equation (27) can be modified as follows,

$$\begin{aligned}\Pi = \sum_{e\in\Omega}\sum_{m=1}^{M} \iint_{\Omega_e} a_m\left|\mathbf{f}_m\right| + \sum_{l=1}^{M}\frac{b_{ml}}{2}\left|\mathbf{f}_m\right|\left|\mathbf{f}_l\right| - \int_0^{q_m} D_m^{-1}(\xi)\mathrm{d}\xi + u_m\left(\nabla\mathbf{f}_m + q_m\right)\mathrm{d}\Omega \\ + \sum_{i=1}^{N_O}\sum_{m=1}^{M}\left(w_{mxi} f_{mxi} + w_{myi} f_{myi}\right) - \sum_{n=1}^{N}\sum_{i=1}^{S_n}\sum_{m=1}^{M} \bar{u}_{mn} L_i \left(\frac{\mathbf{f}_{mj} + \mathbf{f}_{mk}}{2}\right)\cdot \mathbf{n}_i \\ + \sum_{n=1}^{N}\sum_{i=1}^{N_n}\sum_{m=1}^{M} \sigma_i \left(u_{mi} - \bar{u}_{mn}\right)\end{aligned} \tag{31}$$

The three-node linear triangular element is used to approximate the functions of the variables over the solution space. The value of the flow vectors and the cost potential within the element are expressed as follows,

$$f_{mx}(x,y) = N_i(x,y) f_{mxi} + N_j(x,y) f_{mxj} + N_k(x,y) f_{mxk} \tag{32}$$
$$f_{my}(x,y) = N_i(x,y) f_{myi} + N_j(x,y) f_{myj} + N_k(x,y) f_{myk} \tag{33}$$
$$u_m(x,y) = N_i(x,y) u_{mi} + N_j(x,y) u_{mj} + N_k(x,y) u_{mk} \tag{34}$$

where the subscripts i, j, and k represent the three nodes of the triangular element, and the variables with such subscripts are their values at the nodal point. N_i, N_j, and N_k are the linear interpolation functions of the element, and are expressed as follows,

$$N_i(x,y) = \frac{1}{2\Delta}(\alpha_i + \beta_i x + \xi_i y) \tag{35}$$

$$N_j(x,y) = \frac{1}{2\Delta}(\alpha_j + \beta_j x + \xi_j y) \tag{36}$$

$$N_k(x,y) = \frac{1}{2\Delta}(\alpha_k + \beta_k x + \xi_k y) \tag{37}$$

The values of α, β, ξ, and Δ are explicitly related to the nodal co-ordinates (x_i , y_i), (x_j , y_j), and (x_k , y_k) at nodes i, j, and k, respectively, of the triangular element. The explicit forms of α, β, ξ, and Δ are $\alpha_i = x_j y_k - x_k y_j$, $\beta_i = y_j - y_k$, $\xi_i = x_k - x_j$, and $\Delta = (\beta_i\xi_j - \beta_j\xi_i)/2$. The variations of the flow vector and cost potential over the problem domain are expressed as nodal values of the generated triangular elements and the interpolation functions. Substituting Equations (32) to (37) into equation (31) and taking integration, the Lagrangian can now be expressed as a function of all nodal variables, that is $\overline{\Pi}(\mathbf{\Psi})$, where $\mathbf{\Psi} = \mathrm{Col}(\mathbf{f}, \mathbf{u}, \mathbf{w}, \boldsymbol{\sigma})$ and

$$\mathbf{f} = \mathrm{Col}\left(f_{mxi}, f_{myi}, m = 1,2,\ldots,M, i = 1,2,\ldots,M_N\right) \tag{38}$$

$$\mathbf{u} = \mathrm{Col}\left(w_{mi}, m = 1,2,\ldots,M, i = 1,2,\ldots,M_N\right) \tag{39}$$

$$\mathbf{w} = \mathrm{Col}\left(w_{mi}, m = 1,2,\ldots,M, i = 1,2,\ldots,N_O\right) \tag{40}$$

$$\boldsymbol{\sigma} = \mathrm{Col}\left(\sigma_{mi}, m = 1,2,\ldots,M, i = 1,2,\ldots,M_C\right) \tag{41}$$

where M_N, N_O, and M_C are, respectively, the total number of node, number of node on the city boundary and the number of nodes on the facilities' boundary. "Col" represents a column vector. To solve the problem, the stationary point of the Lagrangian should be considered, and is found by the Newtonian algorithm as follows. Let $\mathbf{\Psi}^0$ be an approximate solution to the problem and by means of Taylor's expansion around the point $\mathbf{\Psi}^0$, and we have

$$\Pi(\mathbf{\Psi}) \cong \Pi\left(\mathbf{\Psi}^0\right) + \mathbf{R}^T\left(\mathbf{\Psi} - \mathbf{\Psi}^0\right) + \frac{1}{2}\left(\mathbf{\Psi} - \mathbf{\Psi}^0\right)^T \mathbf{J}\left(\mathbf{\Psi} - \mathbf{\Psi}^0\right) \tag{42}$$

where $\mathbf{R} = \nabla\Pi\left(\mathbf{\Psi}^0\right)$ is the residual vector of the first derivatives evaluate at $\mathbf{\Psi}^0$, and $\mathbf{J} = \nabla^2\Pi\left(\mathbf{\Psi}^0\right)$ is the Jacobian matrix evaluate at $\mathbf{\Psi}^0$. For the stationary point the derivatives of Π with respect to all variables vanish. Therefore,

$$\nabla\Pi = \mathbf{R} + \mathbf{J}\left(\mathbf{\Psi} - \mathbf{\Psi}^0\right) = 0 \tag{43}$$

After rearranging, a better solution can be obtained by

$$\mathbf{\Psi} = \mathbf{\Psi}^0 - \mathbf{J}^{-1}\mathbf{R} \tag{44}$$

Therefore, an iterative procedure can be derived as follows.

Solution Procedure:
Step 1: Find an initial solution $\mathbf{\Psi}^{(0)}$. Set $k = 0$.
Step 2: Evaluate $\mathbf{R}\left(\mathbf{\Psi}^{(k)}\right)$ and $\mathbf{J}\left(\mathbf{\Psi}^{(k)}\right)$.
Step 3: If $\left|\mathbf{R}\left(\mathbf{\Psi}^{(k)}\right)\right| < \varepsilon$, and an acceptable error, then stop and $\mathbf{\Psi}^{(k)}$ is the solution
Step 4: Otherwise, find $\mathbf{\Psi}^{(k+1)} = \mathbf{\Psi}^{(k)} - \mathbf{J}\left(\mathbf{\Psi}^{(k)}\right)^{-1}\mathbf{R}\left(\mathbf{\Psi}^{(k)}\right)$.
Step 5: Replace $\mathbf{\Psi}^{(k)}$ by $\mathbf{\Psi}^{(k+1)}$. Set $k = k + 1$ and go to Step 2.

The total cost consumed by each class of customers in the city, which is a useful performance measures of the system, can be determined by

$$T_m = \sum_{e\in\Omega} \iint_{\Omega_e} u_m D_m(u_m)\,\mathrm{d}\Omega,\; m = 1, \ldots, M \tag{45}$$

For the case of elastic demand, another performance measure known as the social welfare (or net economic benefit) can also be evaluated by

$$S = \sum_{m=1}^{2} \sum_{e\in\Omega} \iint_{\Omega_e} \int_{u_m}^{\infty} D_m(\xi)\,\mathrm{d}\xi\,\mathrm{d}\Omega \tag{46}$$

5. NUMERICAL EXAMPLES

5.1 Example 1: Fixed Demand

In this paper, we present two hypothetical examples to demonstrate the methodology. In Example 1, we consider a modeled city with two facilities as shown in Figure 1. The finite element mesh that is used for analysis is shown in Figure 2. There are two classes of customers, each of which has distinct demand and cost functions. Different classes can be used to represent the heterogeneous characteristics of customers, such as the different values of times for low and high income groups or the different mobility of light and heavy vehicles. The travel demands are fixed at 150 and 100 veh/h/km^2 for class 1 and 2, respectively. The total demand (classes 1 and 2 combined) in the system is 147,779 veh/h. The cost-flow functions are given as

$$c_1 = 0.01 + 0.5\times10^{-4}\left|\mathbf{f}_1\right| + 0.4\times10^{-4}\left|\mathbf{f}_2\right| \tag{47}$$

$$c_2 = 0.05 + 0.4\times10^{-4}\left|\mathbf{f}_1\right| + 0.6\times10^{-4}\left|\mathbf{f}_2\right| \tag{48}$$

throughout the city, where c_1 and c_2 are expressed in terms of h/km. The in-situ costs for class 1 customers are 2.0 and 4.0 hours for facilities 1 and 2, respectively, and those for class 2 customers are 2.5 and 3.5 hours. The model can be solved by the Newtonian algorithm with an acceptable error ε of 10^{-3}. The results of flow vectors for class 1 and class 2 customers are shown in Figures 3 and 4, respectively. These figures show the paths that are chosen by the customers from their home locations to their chosen facilities. The catchment areas of the two facilities can also be visualized from the figures. The flow intensities and generalized costs of the two customers classes are shown in Figures 5 to 8. Note that the unit for flow intensity is veh/h/km, and that for the generalized cost is hours. The total generalized cost incurred by all customers is 672,679 veh-h, with 390,111 and 282,568 veh-h for class 1 and class 2 customers, respectively.

The market shares of facility 1 for class 1 and class 2 customers are 70.8% and 47.9%, respectively, whereas those of facility 2 are 29.2% and 52.1%. For class 1 customers, facility 1 takes up a larger proportion of demand as its facility in-situ cost for class 1 customers is much less than that of facility 2, which can compensate for the extra congestion that is incurred by the customers around that facility. However, for class 2 customers, the situation is the reverse. Although the in-situ cost of facility 1 for class 2 customers is less than that of facility 2, more customers choose to use facility 2 because the congestion cost that is incurred by class 1 customers around facility 1 is relatively large, and cannot be compensated for by the lower in-situ cost at facility 1. Consequently, class 2 customers choose to use a more expensive but less congested facility. This result demonstrates how customers of different classes affect each other in their choices of route in the road network and facility.

5.2 Example 2: Elastic Demand

The elastic demand for the two customer classes is also considered. The demand functions for customer classes 1 and 2 are taken as follows:

$$D_1(u_1) = 400e^{-0.3u_1} \tag{49}$$

$$D_2(u_2) = 350e^{-0.2u_2} \tag{50}$$

where u_1 and u_2 are, respectively, the generalized cost potentials for class 1 and class 2 customers at their home locations. The model can also be solved by the Newtonian algorithm, which gives similar patterns of flow vectors, flow intensities, and generalized costs as for the case of fixed demand. The distribution patterns of demand for class 1 and class 2 customers are shown in Figures 9 and 10, respectively. Generally, the demand intensity decreases with the distance away from the facilities due to increasing generalized cost. The total demand (classes 1 and 2 combined) is 146,783 veh/h for this elastic demand case. The total generalized cost incurred by all customers is 659,986 veh-h, with 279,544 and 380,442 veh-h for class 1 and class 2 customers, respectively. The social welfare for the transportation system is 626,919 veh-h.

6. CONCLUSIONS

The continuum approximation of network flow by Wong and Yang (1999) and Yang and Wong (2000) for a city with a very dense transportation network and multi-competing facilities has been extended to deal with the interactions of multi-class users. In this study, each class of users is assumed to have its own transportation cost function, demand function, and facility in-situ costs. Each user class affects the others through congestion externality in the transportation system. A mathematical program for this multi-class problem has been formulated, and the user equilibrium conditions have been satisfied. The finite element method has been used to discretize the problem, and a Newtonian algorithm has been developed to determine the solution. Both cases of fixed and elastic demand have been used as numerical examples to demonstrate the effectiveness of the proposed methodology. There are two directions for further research. The first is to develop a multi-class model with asymmetric cost functions. The second is to consider the elastic market externality function to account for the congestion effect at the facilities.

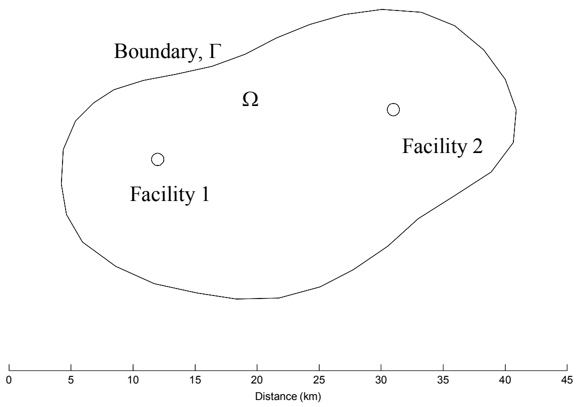

Figure 1: The Modeled City

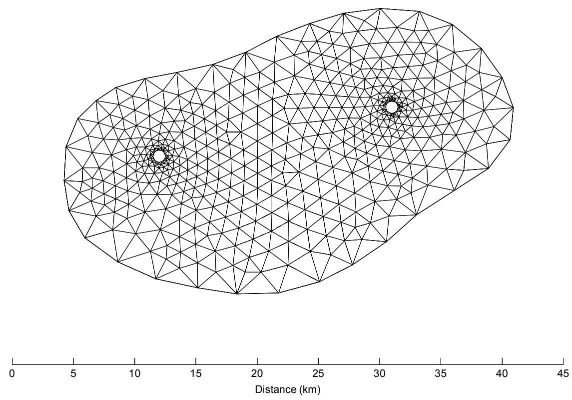

Figure 2: The Finite Element Mesh

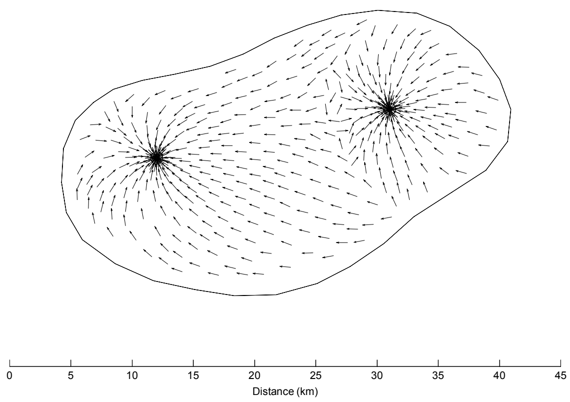

Figure 3: Flow Pattern of Class 1Customers

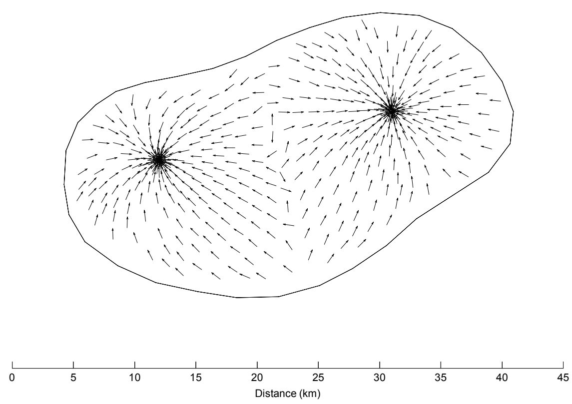

Figure 4: Flow Pattern of Class 2Customers

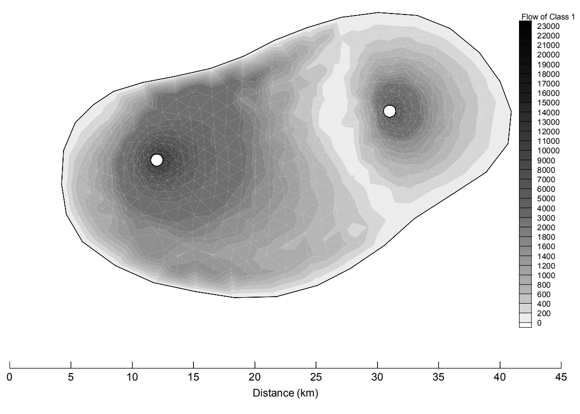

Figure 5: Flow Intensity of Class 1 Customers

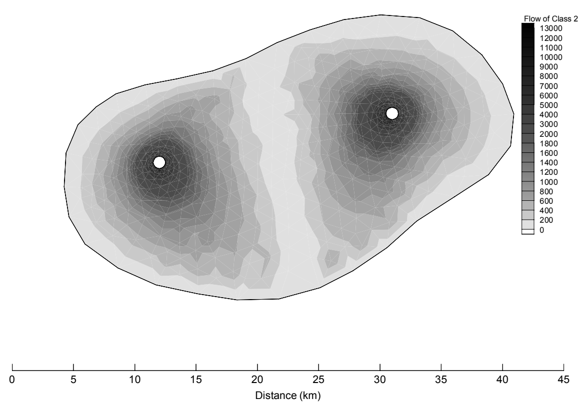

Figure 6: Flow Intensity of Class 2 Customers

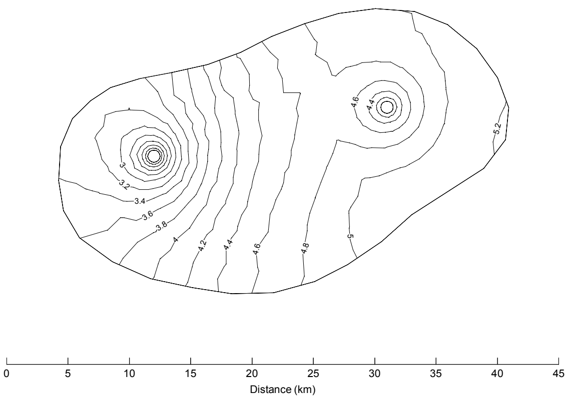

Figure 7: Total Cost for Class 1 Customers

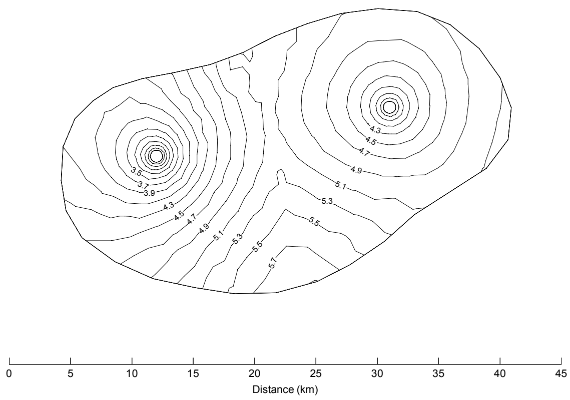

Figure 8: Total Cost for Class 2 Customers

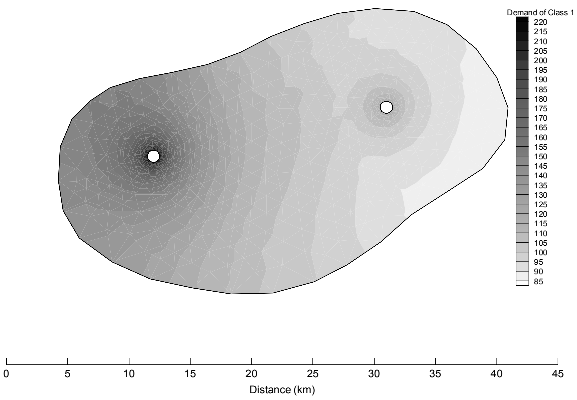

Figure 9: Demand of Class 1 Customers (Elastic Demand Case)

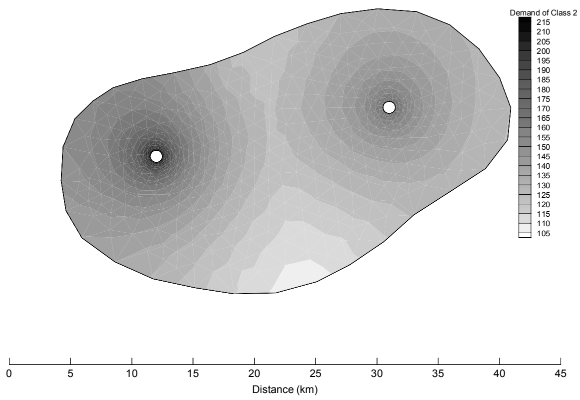

Figure 10: Demand of Class 2 Customers (Elastic Demand Case)

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REFERENCES

Beckmann, M.J. (1952) A continuous model of transportation, **Econometrica, Vol. 20,** 643-666.

Buckley, D.J. (1979) Traffic assignment in a two-dimensional continuous representation of a traffic network, **Transportation Research, Vol. 13B,** 167-179.

Cheung, Y.K., Lo, S.H. and Leung, A.Y.T. (1996) **Finite Element Implementation.** Blackwell Science.

Dafermos, S.C. (1980) Continuum modeling of transportation network, **Transportation Research, Vol. 14B,** 295-301.

D'Este, G. (1987) Trip assignment to radial major roads, **Transportation Research, Vol. 21B**, 433-442.

Ho, H.W., Wong, S.C. and Loo, B.P.Y. (2002) Application of continuum modeling for airports coordination, in D. Wang and S.M. Li (eds.), **Proceedings of the Seventh Conference of Hong Kong Society for Transportation Studies**, Hong Kong, 14 December 2002, 378-380.

Sasaki, T., Iida, Y. and Yang, H. (1990) User-Equilibrium Traffic Assignment by Continuum Approximation of Network Flow, in M. Koshi (ed.), **Proceedings of the 11th International Symposium on Transportation and Traffic Theory,** Yokohama, Japan, July 1990, Elsevier, 233-252.

Vaughan, R.J. (1987) **Urban Spatial Traffic Patterns**. Pion, London.

Vliet, D.V., Bergman, T. and Scheltes, W.H. (1986) Equilibrium Traffic Assignment with Multiple User Classes, **PTRC Summer Annual Meeting, Planning and Transport Research and Computation Co.,** 111-121, England, 14-17, July 1986.

Wardrop, J.G. (1952) Some theoretical aspects of road traffic research, **Proceedings of the Institute of Civil Engineers, Part II**, 325-378.

Williams, H.C.W.L. and Ortuzar Salas, J. D. (1976) Some generalizations and application of the velocity field concept: Trip patterns in idealized cities, **Transportation Research, Vol. 10,** 65-73.

Wong, S.C. (1994) An alternative formulation of D'Este's trip assignment model, **Transportation Research, Vol. 28B**, 187-196.

Wong, S.C. (1998) Multi-commodity traffic assignment by continuum approximation of network flow with variable demand, **Transportation Research, Vol. 32B,** 567-581.

Wong, S.C., Du, Y.C., Ho, H.W. and L.J. Sun (2002a) A simultaneous optimization formulation of a discrete/continuous transportation system, **Journal of Transportation Research Board: Transportation Research Record,** accepted for publication.

Wong, S.C., Lee, C.K. and Tong, C.O. (1998) Finite element solution for the continuum traffic equilibrium problems, **International Journal for Numerical Methods in Engineering, Vol. 43,** 1253-1273.

Wong, S.C. and Sun, S.H. (2001) A combined distribution and assignment model for continuous facility location problem, **The Annals of Regional Science, Vol. 35**, 267-281

Wong, S.C. and Yang, H. (1999) Determining Market Areas Captured by Competitive Facilities: A Continuous Equilibrium Modeling Approach, **Journal of Regional Science, Vol. 39, No. 1,** 51-72.

Wong, S.C., Zhou, C.W., Lo, H.K. and Yang, H. (2002b) An improved solution algorithm for the multi-commodity continuous distribution and assignment model, **ASCE Journal of Urban Planning and Development**, accepted for publication.

Yang, H., Yagar, S. and Iida, Y. (1994) Traffic assignment in a congested discrete/continuous transportation system, **Transportation Research, Vol. 28B**, 161-174.

Yang, H. and Wong, S.C. (2000) A Continuous Equilibrium Model for Estimating Market Areas of Competitive Facilities with Elastic Demand and Market Externality, **Transportation Science, Vol. 34, No. 2,** 216-227.

Zienkiewicz, O.C. and Taylor, R.L. (1989) **The Finite Element Method,** McGraw-Hill, International Edition.

Zitron, N. R. (1974) A Continuous Model of Optimal-Cost Routes in a Circular City, **Journal of Optimization Theory and Applications, Vol. 14, No. 3,** 291-303.